AP® Calculus BC Practice Exam

Instructions:

Write your solution using dark pencil or ink on white paper. Label each part of each question. Write clearly and legibly. Cross out any errors you make; erased or crossed-out work will not be scored.

Manage your time carefully. During the timed portion for Part A, work only on the questions in Part A. Calculators are permitted but are not required for any part of any question. If using a calculator, you must clearly indicate your setup, and you must show the mathematical steps necessary to produce your results. Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt($X^2, X, 1, 5$).

Show all of your work, even though a question may not explicitly remind you to do so. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If you use decimal approximations in calculations, your work will be scored on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

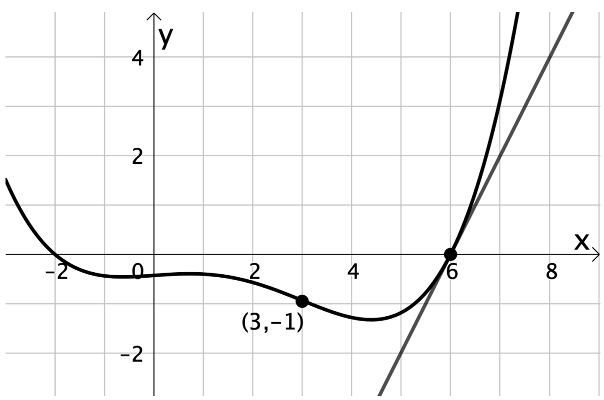
CALCULUS BC

Part A

Time—25 minutes

Number of problems—1

A GRAPHING CALCULATOR IS NOT REQUIRED FOR THESE PROBLEMS.



Graph of f

x	1	2	3	4	5	6
$f''(x)$	$-\frac{1}{2}$	1	4	8	4	3
$f'''(x)$	$\frac{1}{2}$	2	3	6	-1	-2

1. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 6$. Selected derivatives of f are given in the table above.

   (a) Let $g(x) = \int_3^x f(2t)\,dt$. Write the fourth-degree Taylor polynomial for g about $x = 3$.

Solution:

$g^{(n)}(x)$	$g^{(n)}(3)$
$g(x) = \int_3^x f(2t)\,dt$	$g(3) = \int_3^x 3f(2t)\,dt = 0$
$g'(x) = f(2x)$	$g'(3) = f(6) = 0$
$g''(x) = 2f'(2x)$	$g''(3) = 2f'(6) = 4$
$g'''(x) = 4f''(2x)$	$g'''(3) = 4f''(6) = 12$
$g^{(4)}(x) = 8f'''(2x)$	$g^{(4)}(3) = 8f'''(6) = -16$

$$P_4(x) = g(3) + g'(3)(x-3) + \frac{g''(3)}{2!}(x-3)^2 + \frac{g'''(3)}{3!}(x-3)^3 + \frac{g^{(4)}(3)}{4!}(x-3)^4$$

$$= \frac{4}{2!}(x-3)^2 + \frac{12}{3!}(x-3)^3 - \frac{16}{4!}(x-3)^4$$

Score:
1: chain rule
1: first term
1: remaining terms

(b) Find the x-coordinate of each critical point of g on the interval $-2.5 \leq x \leq 7$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Solution:
Let $g'(x) = f(2x) = 0$. On the interval $-2.5 \leq x \leq 7$, $f(-2) = f(6) = 0$. Therefore $2x = \{-2, 6\}$, which gives $x = \{-1, 3\}$.

At $x = -1$, g' is changing from positive to negative, therefore g has a relative maximum at $x = -1$.

At $x = 3$, g' is changing from negative to postive, therefore g has a relative minimum at $x = -1$.

Score:
1: critical values at $x = \{-1, 3\}$
1: relative max at $x = -1$, with justification
1: relative min at $x = 3$, with justification

(c) Given $f'(3) = -\frac{1}{2}$, evaluate $\int_{1.5}^{3} x^2 g^{(4)}(x)\, dx$.

Solution:

$$\begin{aligned}
\int_{1.5}^{3} x^2 g^{(4)}(x)\, dx &= [x^2 g'''(x)]_{1.5}^{3} - \int_{1.5}^{3} (2x) g'''(x)\, dx \\
&= [(x^2)(4) f''(2x)]_{1.5}^{3} - [(2x) g''(x)]_{1.5}^{3} + \int_{1.5}^{3} 2g''(x)\, dx \\
&= 36 f''(6) - 9 f''(3) - [4x f'(2x)]_{1.5}^{3} + 2[g'(x)]_{1.5}^{3} \\
&= 108 - 36 - [12 f'(6) - 6 f'(3)] + 2[f(2x)]_{1.5}^{3} \\
&= 108 - 36 - 24 + 6 f'(3) + 2 f(6) - 2 f(3) \\
&= 108 - 36 - 24 - 3 + 2 \\
&= 47
\end{aligned}$$

Score:

1: integration by parts

1: answer

(d) Let $\sum_{n=1}^{\infty} a_n$ be a divergent geometric series. Does $\sum_{n=1}^{\infty} \frac{a_n}{n!}$ converge or diverge? Explain.

Solution:

Since $\sum_{n=1}^{\infty} a_n$ is a geometric series, we know $\frac{a_{n+1}}{a_n} = r$.

We will use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{a_n}{n!}$ converges or diverges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{(n+1)} \cdot \frac{1}{a_n} \right| = \lim_{n\to\infty} \left| \frac{r}{n+1} \right| = 0 < 1$$

Therefore, $\sum_{n=1}^{\infty} \frac{a_n}{n!}$ converges by the ratio test.

Score:

1: ratio test setup correctly

1: limit

1: conclusion

(e) Let $\sum_{n=1}^{\infty} b_n$ be the sequence defined by the following recursive formula:

$$b_1 = 1;\ b_{n+1} = \left(1 + \frac{1}{n}\right)^n b_n.$$

i. Find the third partial sum of the series $\sum_{n=1}^{\infty} b_n$.

ii. Use the ratio test to determine whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.

Solution:

$$\begin{aligned} S_3 &= 1 + \left(1 + \frac{1}{1}\right)^1 (1) + \left(1 + \frac{1}{2}\right)^2 (2) \\ &= 1 + 2 + \frac{9}{2} \\ &= 7.5 \end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| &= \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^n b_n}{b_n}\right| \\
&= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n\right| \\
&= e \\
&> 1
\end{aligned}
$$

Therefore $\sum_{n=1}^{\infty} b_n$ diverges.

Score:

1: $S_3 = 7.5$

1: ratio test

1: conclusion

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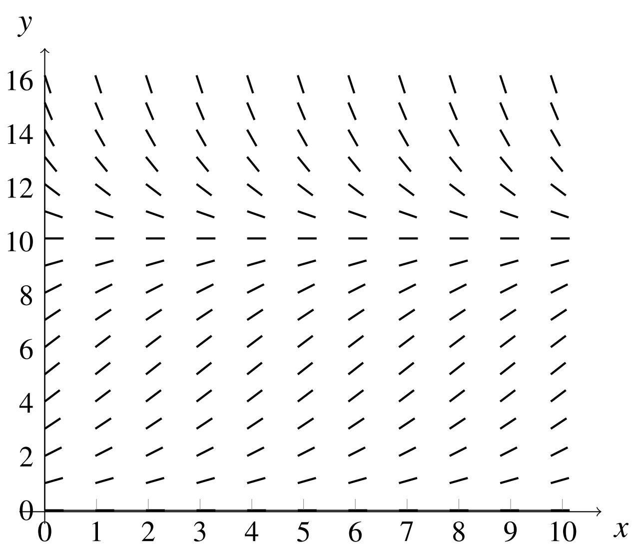
Part B

Time—15 minutes

Number of problems—1

A GRAPHING CALCULATOR IS NOT REQUIRED FOR THESE PROBLEMS.



2. The differential equation $\frac{dy}{dx} = ky(M - y)$ has the slope field shown above. Let $y = f(x)$ be the particular solution to the given differential equation with $f(0) = 1$.

(a) Use Euler's method, with two steps of equal size, to approximate $f(2)$ in terms of k.

Solution:

x	y	$y' = ky(10 - y)$
0	1	$9k$
1	$1 + 9k$	$k(1 + 9k)(9 - 9k)$
2	$(1 + 9k) + k(1 + 9k)(9 - 9k)$	

$$\begin{aligned} f(2) &\approx (1 + 9k) + k(1 + 9k)(9 - 9k) \\ &= (1 + 9k)(1 + k(9 - 9k)) \\ &= -(9k + 1)(9k^2 - 9k - 1) \end{aligned}$$

Score:
1: Euler's method
1: answer

(b) Find $f''(0)$ in terms of k and y.

Solution:

$$\begin{aligned} \left.\frac{d^2y}{dx^2}\right|_{(0,1)} &= k\frac{dy}{dx}(10-y) + k\left(-\frac{dy}{dx}\right) \\ &= k(9k)(10-1) + k(-9k) \\ &= 81k^2 - 9k^2 \\ &= 72k^2 \end{aligned}$$

Score:
1: derivative
1: answer

(c) The actual value of $f(2) < 4.6$. Is your answer to part (a) an underestimate or overestimate? Justify your response.

Solution:
The graph of f changes from concave up to concave down where $f(x) = 5$. Since $f(2) < 5$, we can conclude that the graph of f is concave up on $[0, 2]$. Hence, Euler's method results in an underestimate.

Score:
1: underestimate
1: reason

(d) Let $y = g(x)$ be the particular solution to the given differential equation with $g(1) = 12$. Find $\lim\limits_{x\to\infty}\left(\frac{g(x)-10}{f(x)-10}\right)$, or explain why it does not exist.

Solution:
$\lim\limits_{x\to\infty}(g(x) - 10) = 0$

$\lim\limits_{x\to\infty}(f(x) - 10) = 0$

Therefore, L'Hôpital's rule may be used.

$$\lim_{x\to\infty} \frac{g(x) - 10}{f(x) - 10} = \lim_{x\to\infty} \frac{g'(x)}{f'(x)} = \frac{ky(10-y)}{ky(10-y)} = 1$$

Score:

1: conditions of L'Hôpital's rule verified
1: L'Hôpital's rule
1: answer

Suggested Scoring:

Raw Score:	Exam Score:
14-23	5
12-13	4
9-11	3
6-8	2
0-5	1