Topics Covered AP Calculus AB so far

1) Elementary Functions

a) <u>Properties of Functions</u>

i) A <u>function</u> *f* is defined as a set of all ordered pairs (*x*, *y*), such that for each element *x*, there corresponds exactly one element *y*. The <u>domain</u> of *f* is the set *x* (all possible *x* values). The <u>range</u> of *f* is the set of *y* (all possible *y* values).

b) <u>Inverse functions</u>

i) Functions *f* and *g* are inverses of each other if

(1) $f(g(x)) = x$

(2) **And** $g(f(x)) = x$

ii) To find $f^{-1}(x)$, switch *x* and *y* in the original equation and solve the equation for *y* in terms of *x*.

c) <u>Zeros of a Function</u>

i) We also call these the <u>roots</u> or the <u>*x* intercepts</u>. These occur where the function *f*(x) crosses the *x*-axis.

ii) To find the zeros of a function, set *y* equal to zero and solve for *x*.

2) Limits

a) $\lim_{x \to c} f(x) = L$ ("the limit of *f*(*x*) as *x* approaches c is L)

b) <u>Properties of Limits</u>

i) Scalar multiple: $\lim_{x\to c}\left[b(f(x))\right] = b\lim_{x\to c}\left[(f(x))\right]$

ii) Sum or Difference: $\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

iii) Product: $\lim_{x\to c}\left[f(x)\cdot g(x)\right] = \left[\lim_{x\to c} f(x)\right]\cdot\left[\lim_{x\to c} g(x)\right]$

iv) Quotient: $\lim_{x\to c}\left[f(x)/g(x)\right] = \left[\lim_{x\to c} f(x)\right]\Big/\left[\lim_{x\to c} g(x)\right]$

c) <u>Limits that Fail to Exist</u>

i) Behavior that different from the right and from the left

(1) EX: $\lim_{x\to 0}\frac{|x|}{x}$

ii) Unbounded behavior – also known as Infinite Limits (when the limit "equals" positive or negative infinity)

(1) EX: $\lim_{x\to 0}\frac{1}{x^2}$

iii) Oscillating behavior

(1) EX: $\lim_{x\to 0}\sin\frac{1}{x}$

d) <u>Special Trig Limits</u>

i) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

ii) $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

e) <u>One-sided Limits</u>

i) $\lim_{x\to a^+} f(x)$ — *x* approaches *c* from the right

ii) $\lim_{x\to a^-} f(x)$ — *x* approaches *c* from the left

f) **<u>Continuity</u>**

i) *<u>Definition</u>*: A function f is continuous at c if:

(1) $f(c)$ is defined (the point exists)

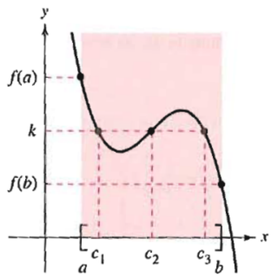
(2) $\lim_{x \to c} f(x)$ exists

(3) $\lim_{x \to c} f(x) = f(c)$

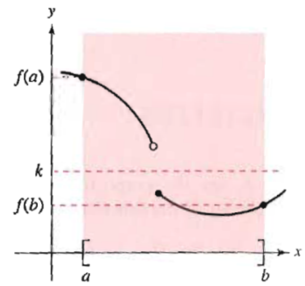
ii) Graphically, we can see if a function is continuous if a pencil can be moved along the graph of $f(x)$ without having to be lifted off the paper.

g) **<u>IVT (Intermediate Value Theorem) (Section 1.4)</u>**

i) If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c) = k$.



f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]

ii) We used this theorem to prove that a zero existed on a closed interval.

3) Differential Calculus

a) **<u>Definition</u>**:

i) $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ (general derivative function) **<u>and</u>**

$f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ (derivative at a specific point, c), provided that these limits exist.

ii) Remember, the derivative represents the **slope** of a function at a specific point.

iii) We can use the derivative to find the **slope of the line that is tangent** to the function at a specific point. To find the equation of a tangent line, first take the derivative and find the slope at the given point. Then using that point and the slope you just found, write the equation of the tangent line in point-slope form: $y - y_1 = m(x - x_1)$.

b) **<u>Differentiation Rules</u>**

i) *<u>General and Logarithmic Differentiation Rules</u>*

1. $\dfrac{d}{dx}[cu] = cu'$
2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$ sum rule
3. $\dfrac{d}{dx}[uv] = uv' + vu'$ product rule
4. $\dfrac{d}{dx}[\dfrac{u}{v}] = \dfrac{vu' - uv'}{v^2}$ quotient rule
5. $\dfrac{d}{dx}[c] = 0$
6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$ power rule
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
9. $\dfrac{d}{dx}[e^u] = e^u u'$
10. $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$ chain rule

ii) *Derivatives of Trigonometric Functions*

1. $\frac{d}{dx}[\sin u] = (\cos u)u'$
2. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
3. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
4. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
5. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
6. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

The most important trig identity to remember is the Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$
just divide by sin^2 x or cos^2 x to get the other functions.

c) Implicit Differentiation

i) Use implicit differentiation when you have x's and y's mixed together and you cannot separate them.

ii) EX: To find $\frac{dy}{dx}$ of $y^3 + xy - 2y - x^2 = -2$, we must use implicit differentiation

(1) Take the derivative of each term (don't forget to use product rule!)

(2) Any time you take the derivative of a y term, include $\frac{dy}{dx}$.

(3) Separate terms so that all $\frac{dy}{dx}$ terms are on one side of the equal sign and everything else is on the other side.

(4) Solve for $\frac{dy}{dx}$.

d) Higher Order Derivatives

i) You can take the derivative of a function multiple times. You can have the first, second, third . . . and nth derivative of a function.

ii) We used this to talk about **position, velocity and acceleration**

(1) $s(t)$ The *position* function

(2) $v(t) = s'(t)$The *velocity* function is the first derivative of the position function

(3) $a(t) = v'(t) = s''(t)$The *acceleration* function is the first derivative of the velocity function, or the second derivative of the position function.

Word Problems

(Notes by Michael Samra)

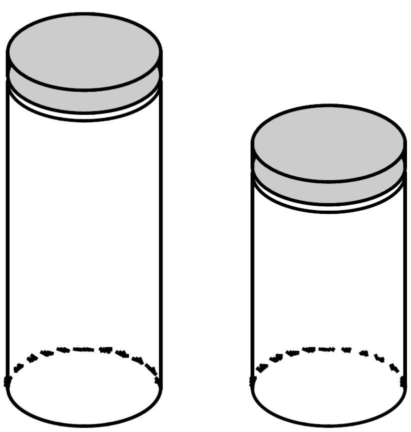
The two main types of word problems introduced in the first semester of Calculus are **related rates** problems and **optimization (or maximum-minimum)** problems. Each are briefly explained below.

Related rates problems have to do with quantities or variables that are varying in time but that always satisfy some relation. For example: suppose the radius of a circle is increasing at the constant rate of 2 inches per second. How fast is the area of the circle increasing when the radius is 10 inches? You imagine the circle expanding at a constant rate, and, at the instant when the radius is 10 inches, you are asked to determine the rate at which the area of the circle is increasing. Here, the equation that is always satisfied between the area and radius of a circle is $A = \pi r^2$.

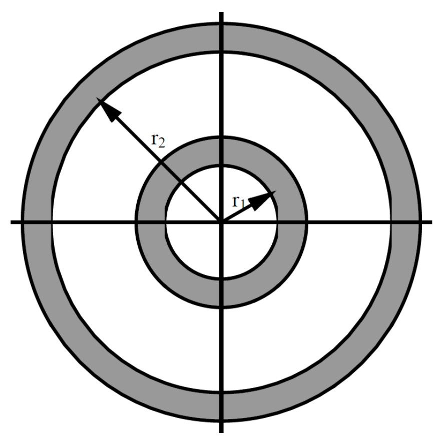
A similar example is the following: how fast is the volume of a cylinder of fixed radius 4 inches increasing, given that the height is increasing at the rate of 2 inches per second, at the moment in time when the height is 10 inches?

Related rates questions usually ask you to find the rate of change of one variable given the rate of change of the other related variable or variables at some instant of time, usually specified by a value of one of the variables.

To contrast the above two examples, the constant increase of the height of the cylinder produces a constant increase of its volume, whereas the constant increase of the radius of the circle doesn't produce a constant increase of its area: the area increases faster at larger values for the radius r (see the diagram below). Can you understand why this is true in terms of the equations for the volume of a cylinder and the area of a circle?



At any height, the same increase in height adds the same volume to the cylinder.



By comparing, for instance, the shaded regions in Quadrant I, you can see that the same increase in the radius produces a larger increase in the area at the larger radius r_2.

Related Rates Examples

The first example will be used to give a general understanding of related rates problems, while the specific steps will be given in the next example.

Example 1: The radius of a circle is increasing at the rate of 2 inches per second. What is the rate of change of the area when the radius is 10 inches?

This problem demonstrates the basic idea of related rates. There are quantities that are varying with time (the radius and the area are increasing as the circle gets bigger), but that always satisfy some equation ($A = \pi r^2$). The question is usually to find the rate of change of one of the variables given the rate of change of the other variable(s), at some moment specified by a value for one (or more) of the variables.

The equation that A and r satisfy is more fully written as $A(t) = \pi(r(t))^2$ to show that A and r are functions of time t. It is customary, though, to simply write $A = \pi r^2$ and apply $\frac{d}{dt}$ to both sides of the equation - to implicitly differentiate the equation with respect to time:

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Replacing r with 10 and $\frac{dr}{dt}$ with 2, gives the answer that the area is increasing at the rate of 40π square inches per second when the radius is 10 inches.

As mentioned above, the rate of change of the area increases as the radius gets larger. From the equation

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt},$$

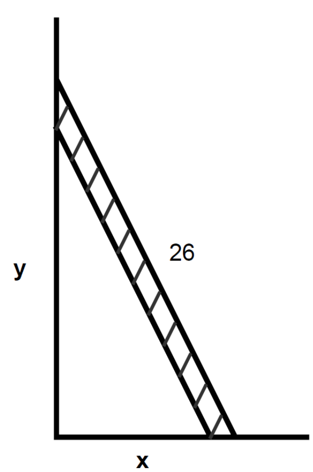
$\frac{dr}{dt}$ can be replaced by the constant 2, to give the equation

$$\frac{dA}{dt} = 4\pi r.$$

This equation shows that the rate of change of the area of the circle is greater at larger values of the radius.

Example 2: A ladder 26 feet long leans against a vertical wall. If the bottom of the ladder is being drawn away from the wall at the rate of 4 feet per second, how fast is the ladder moving down the wall when the bottom of the ladder is 10 feet from the wall?

Step 1: *Draw a diagram, if possible, labeling with variables those quantities that vary in time and with numbers those quantities that remain constant.*



A common mistake is to label the quantity that specifies the moment in time with a number rather than a variable. In this problem, the ladder is being drawn away from the wall, so the distance x is varying in time - it is not fixed at 10 feet. As the ladder is drawn away from the wall, it slides down the wall, so y is varying in time. On the other hand, the ladder has the fixed length of 26 feet.

Step 2: *Using mathematical notation, write what's given and what you are asked to find.*

Given $\frac{dx}{dt} = 4\,\frac{\text{ft}}{\text{sec}}$, find $\frac{dy}{dt}$ when $x = 10$ feet.

Step 3: *Write an equation involving the variables in the problem.*

$$x^2 + y^2 = 26^2$$

In many cases, the equation will be an area or volume formula like the previous example. In this example, the Pythagorean Theorem is used. In some other examples, you might have to do an additional step such as using similar triangles to relate two or more of the quantities.

Step 4: *Implicitly differentiate the equation with respect to time.*

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Step 5: *Plug in the given quantities to solve.* (Sometimes it is also necessary to find the value of the other variable(s) at the specified moment in time).

We are given $x = 10$ and $\frac{dx}{dt} = 4$. We need to find what y is when $x = 10$. To find y, we use the Pythagorean Theorem again for the particular moment when $x = 10$:

$$10^2 + y^2 = 26^2 \quad , \qquad y = 24$$

$$10\cdot 4 + 24\cdot\frac{dy}{dt} = 0 \quad , \qquad \frac{dy}{dt} = -\frac{5}{3} \text{ feet per second.}$$

The question asks how *fast* the ladder is moving *down* the wall, so the correct answer is $\frac{5}{3}$ feet per second.