Chap 3 Test Practice
Nov 10 2022 answers

1. Use the Candidates Test to identify the absolute extrema of $f(x) = x^3 - 6x^2 + 9x + 5$ on the interval $0 \le x \le 4$.

$f'(x) = 3x^2 - 12x + 9$ End points: $x=0$ $x=4$

$0 = 3(x-1)(x-3)$ Critical points: $x=1, x=3$

x	0	1	3	4
$f(x)$	5	9	5	9

Ab. max on $[0,4]$ is 9
Ab min on $[0,4]$ is 5

by the Candidates Test

2. Use the Candidates Test to identify the absolute extrema of $f(x) = x + \dfrac{7}{x}$ on the interval $1 \le x \le 3$.

$f'(x) = 1 - \frac{7}{x^2}$

$0 = 1 - \frac{7}{x^2}$

$\frac{7}{x^2} = 1$

$x = \pm\sqrt{7}$ but only one on $[1,3]$

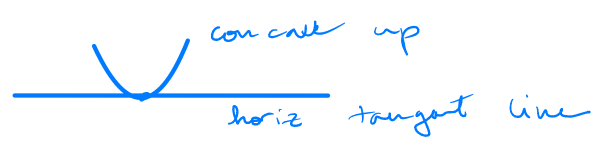
CP: $\sqrt{7}$ (0 is not on $[1,3]$ either)

x	1	$\sqrt{7}$	3
$f(x)$	8	$\sqrt{7}+\frac{7}{\sqrt{7}}$ $(2\sqrt{7})$	$\frac{16}{3}$ $(=5.\bar{3})$

Ab max on $[1,3]$ is 8
Ab min on $[1,3]$ is $\sqrt{7}+\frac{7}{\sqrt{7}}$ or $\sqrt{7}+\frac{7\sqrt{7}}{7}$ or $2\sqrt{7}$

3. Let f be a twice differentiable function. If $f'(7) = 0$ and $f''(7) > 0$ what conclusion can be made and why?

$f(7)$ is a relative min by the 2^{nd} Der Test



4. (Calculator NOT Active) Find at least one c such that the Mean Value Theorem applies to the function $f(x) = x^3$ on the interval $[0, 1]$ and write the equation(s) of the tangent line(s) to the curve at $x = c$.

$$AROC = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1 = f'\left(\frac{\sqrt{3}}{3}\right)$$

Since $f'(x) = 3x^2$

$$1 = 3x^2$$

$$x = \pm\sqrt{\tfrac{1}{3}}\text{, only } \frac{\sqrt{3}}{3} \text{ on } [0, 1]$$

Eq of tangent line:

$$y - \left(\sqrt{\tfrac{1}{3}}\right)^3 = 1\left(x - \sqrt{\tfrac{1}{3}}\right)$$

5. (Calculator Active) Find all c on $[0, 2]$ such that the Mean Value Theorem applies to the function $f(x) = x^2 + 3x - 4\sin(2x + 3)$ on the interval $[0, 2]$.

$$AROC = \frac{f(2) - f(0)}{2 - 0} = 3.968 = f'(0.909)$$

Since $f'(x) = 2x + 3 - 8\cos(2x+3)$

$3.968 = f'(x)$ when x is 0.909

(Tips: make xMin = 0, xMax = 2)
$\frac{f(2) - f(0)}{2 - 0} \rightarrow \boxed{\text{STO}}\ \boxed{\text{ALPHA}}$ A, Make $Y_1 = f'$ & $Y_2 = A$)

6. Find absolute extrema for the function $f(x) = x^3 - 3x + 2$ on the interval $[-3, 2]$. Justify your conclusion.

use Cand. Test

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

CP when $x = \pm 1$

x	-3	-1	1	2
$f(x)$	-16	4	0	4

By Cand. Test, Ab min is -16
Ab max is 4

7. Let $f(x) = x^3 - 3x + 2$.

   (a) Find f' and f'',

   $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

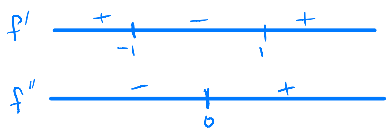
   $f''(x) = 6x$

   (b) Find the critical points of f' and f''.

   for f' : $(-1, 4)$ and $(1, 0)$

   for f'' : $(0, 2)$

   (c) Draw sign lines for f' and f''

   (d) On what intervals are f increasing and decreasing? Justify.

   inc: $(-\infty, -1]$ and $[1, \infty)$

   dec: $[-1, 1]$ (Recall our textbook includes boundries in Th 3.5 p 181)

   (e) On what intervals are f concave up or down? Justify.

   c.c. up on $(0, \infty)$

   c.c. down $(-\infty, 0)$

   (f) Find all relative extrema and justify your conclusions with the 1^{st} Derivative Test

   local min at $(1, 0)$ because f' changes from neg to pos @ $x = 1$

   local max at $(-1, 4)$ because f' changes from pos. to neg @ $x = -1$

   (g) Find all relative extrema and justify your conclusions with the 2^{nd} Derivative Test

   local min at $(1, 0)$ because $f'(1) = 0$ and $f''(1) > 0$

   local max at $(-1, 4)$ because $f'(-1) = 0$ and $f''(-1) < 0$

   (h) If any exist, find any points of inflection (POI). Justify.

   $(0, 2)$ is a P.O.I because f'' changes sign at $x = 0$

   (i) What is the end behavior of f (That is, find $\lim_{x \to \pm\infty} f(x)$)

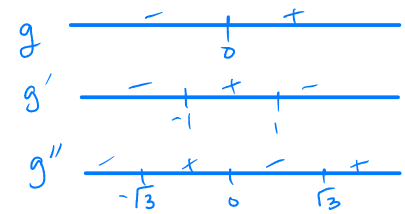
   Because f is a odd degree with a leading coeff

   $$\lim_{x \to -\infty} f(x) = -\infty$$

   $$\lim_{x \to \infty} f(x) = +\infty$$

8. (Calculator NOT active) Given the function $g(x) = \dfrac{4x}{x^2+1}$, its 1^{st} derivative $g'(x) = \dfrac{4(1-x^2)}{(x^2+1)^2}$ and its 2^{nd} derivative $g''(x) = \dfrac{8x(x^2-3)}{(x^2+1)^3}$

(a) Make sign lines for g, g', and g''



(b) Find the (x, y) coordinates of any intercepts, and graph them.

$(0,0)$

(c) Find any vertical or horizontal asymptotes and g's end behavior (i.e.: $\lim\limits_{x\to\pm\infty} g(x)$) and use this to improve the graph.

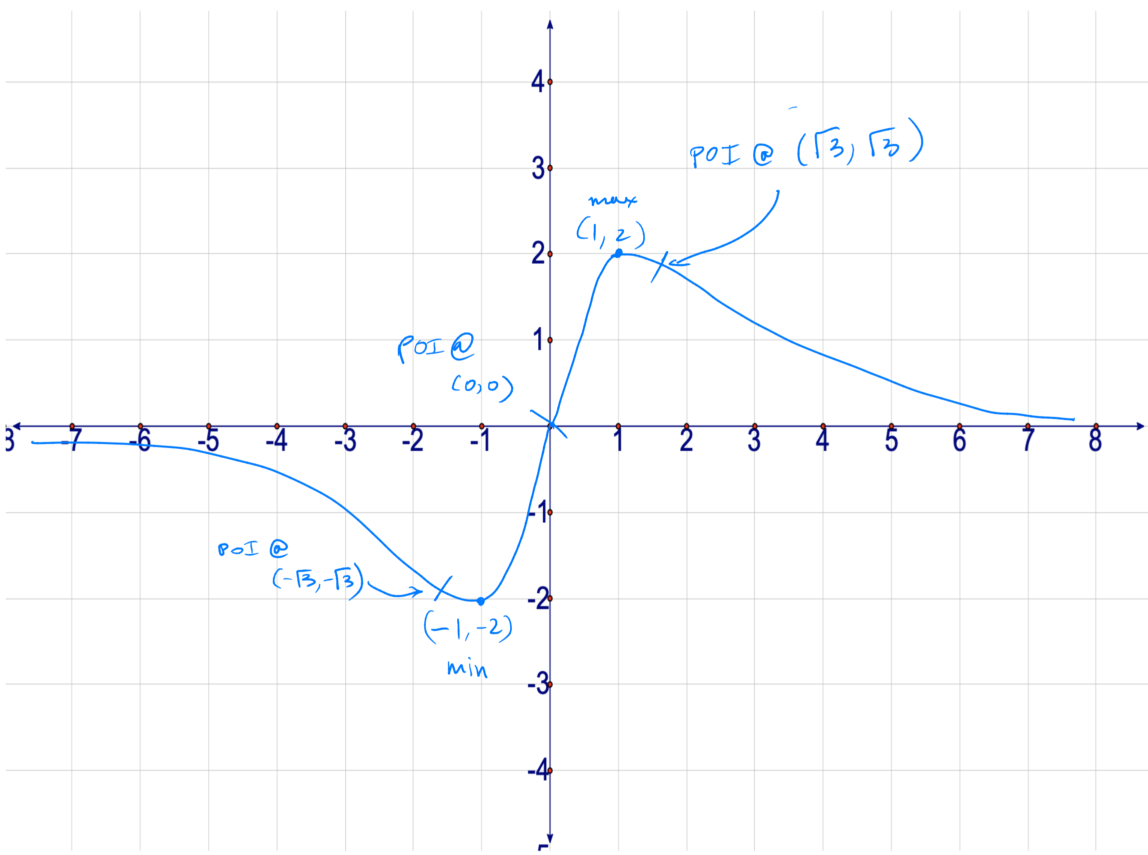
$\left.\begin{matrix} n=1 \\ m=2 \end{matrix}\right\}$ $n<m$ so $\lim\limits_{x\to\pm\infty} f(x) = 0$, horz asymptote is $y=0$

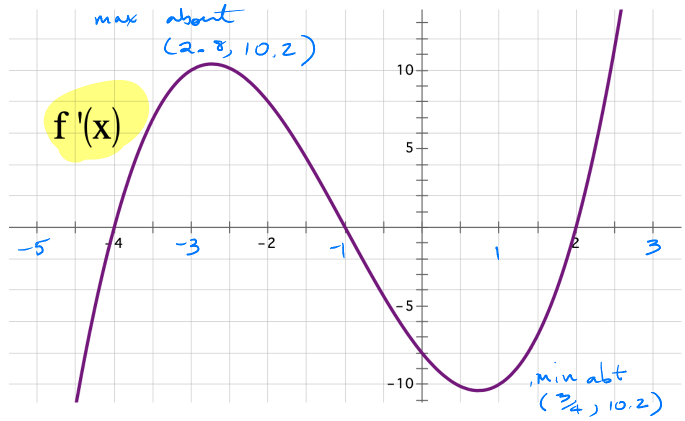
(d) Find the relative extrema and indicate them on the graph with a small horizontal bar through any extrema. $f'(-1)=0$ & $f''(-1)>0$ so $(-1,-2)$ rel. min

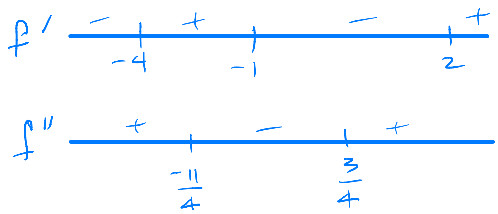
$f'(1)=0$ & $f''(1)<0$ so $(1,2)$ rel. max

(e) Improve the graph by noting where it is concave up or concave down, and find any points of inflection. Indicate a POI on the graph with a small perpendicular bar through the point.

cc up: $(-\sqrt{3},0)$ & $(\sqrt{3},\infty)$ cc down: $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$







9. (Calculator NOT active) Given the graph of $f'(x)$ above:

   (a) Approximate the sign lines for $f'(x)$ and $f''(x)$.

   (b) Find a relative maximum of f and use the 1^{st} Derivative Test to justify your conclusion.

   $f(-1)$ rel max because $f'(x)$ changes from pos to neg at $x=-1$

   (c) Find a relative minimum of f and use the 2^{nd} Derivative Test to justify your conclusion.

   $f(-4)$ and $f(2)$ are rel. min because $f'(-4)=f'(2)=0$ and $f''(-4)>0$ and $f''(2)>0$.

   (d) On what interval(s) is f increasing? Justify.

   when $f'>0$: $[-4,-1]$ and $[2,\infty)$

   (e) On what interval(s) is f decreasing? Justify.

   when $f'<0$: $(-\infty,-4]$, $[-1,2]$

   (f) On what interval(s) is f concave up? Justify.

   when $f''>0$: $(-\infty,-\frac{11}{4})$ and $(\frac{3}{4},\infty)$

   (g) On what interval(s) is f concave down? Justify.

   when $f''<0$: $(-\frac{11}{4},\frac{3}{4})$

   (h) Find all the points of inflection of f, and justify why it is a point of inflection.

   at $(-\frac{11}{4}, f(-\frac{11}{4}))$ and $(\frac{3}{4}, f(\frac{3}{4}))$

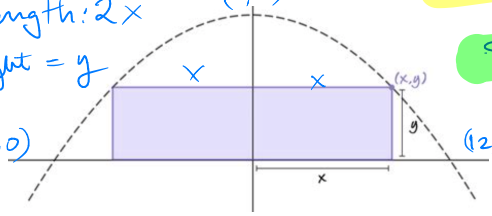
   since f'' changes sign at $x=-\frac{11}{4}$ and $\frac{3}{4}$

10. (Calculator NOT Active) A rectangle has its bottom edge on the x axis and its top corners are on the the graph $y = 6 - \frac{x^2}{24}$ What length and width should the rectangle have so that the rectangle has the largest area?

length: $2x$

height $= y$

$(0,6)$ $(-12,0)$ $(12,0)$



Primary: $A(x,y) = 2xy$ (max)

Sec: $y = 6 - \frac{x^2}{24}$

$A(x) = 2x\left(6 - \frac{x^2}{24}\right)$

$A(x) = 12x - \frac{x^3}{12}$

$A'(x) = 12 - \frac{x^2}{4}$

$A'(x) = 0$ when

$x^2 = 4(12)$

$x = \pm 4\sqrt{3}$

Plausible Domain: $[0, 12]$

Candidate's test

x	0	$4\sqrt{3}$	12
$f(x)$	0	$32\sqrt{3}$	0

$32\sqrt{3}$ is Ab max on Domain ⊛

length $= 2(4\sqrt{3}) = 8\sqrt{3}$

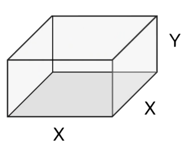
width $= y = 6 - \frac{(4\sqrt{3})^2}{24} = 4$

The max area is when the width is $8\sqrt{3}$ and the length is 4.

(OR)

⊛ Alt: A' $\underset{4\sqrt{3}}{+ \mid -}$ (Rel max @ $x = 4\sqrt{3}$, A' Δ's from pos to neg)

11. (Calculator NOT Active) Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 500 cubic feet of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass?



Primary: $A(x,y) = x^2 + 4xy$ (min)

Sec: $x^2 y = 500$

$y = \frac{500}{x^2}$

$A(x) = x^2 + 4x\left(\frac{500}{x^2}\right)$

$A(x) = x^2 + \frac{2000}{x} = \frac{x^3 + 2000}{x}$

$A'(x) = 2x - \frac{2000}{x^2} = \frac{2(x^3 - 1000)}{x^2}$

Rel. Max at $x = 10$

A' Δ's from pos to neg $\rightarrow$ A' $\underset{0 \qquad 10 \qquad \sqrt{500}}{+ \mid -}$

Ab max @ $x = 10$ by cand. Test $\rightarrow$

x	0	10	$\sqrt{500}$
$A(x)$	0	300	$500 + 40\sqrt{5}$

The least amount of glass is when the base is 10′ by 10′ and the height is 5′

12. In 1977 P.M. Tuchinsky wrote an article called "The Human Cough". In it, the speed s of the air leaving your windpipe as you cough can be modeled by the function

$$s(x) = kx^2(R - x)$$

where R is radius of your windpipe (trachea) while resting (a positive constant), x is the radius of your windpipe while coughing (a positive variable), and k is positive constant.

If for a particular person $k = \frac{1}{3}$ and $R = 27$ mm, find the absolute maximum speed on the interval $0 \le x \le 27$. (Your answer will be in terms of mm per second)

$S(x) = \frac{x^2}{3}(27 - x) = 9x^2 - \frac{x^3}{3}$

$S'(x) = 18x - x^2 = x(18 - x)$

x	0	18	27
$S(x)$	0	972	0

by Cand Test $x = 18$ Ab max is 972

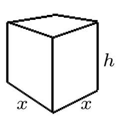
OR

S': $-$ 0 $+$ 18 $-$

by 1st Der s' Δ's sign from pos to neg at $x = 18$, local max

The max speed of air leaving a cough is 972 mm per second

13. A box has 2 square ends and 4 rectangular sides. The square ends are made out of plastic that costs \$5 dollars per square foot, and the cardboard sides cost \$3 dollars per square foot. Find the dimensions of the box that has a volume of 60 cubic feet, that is the cheapest to make. *Hint:* Use the candidates test, first derivative test, or second derivative test to justify your claim that it is indeed the minimum cost.



Primary $C(x, h) = \$5(2x^2) + \$3(4xh)$

Secondary $V = 60 = x^2 h$, $h = \frac{60}{x^2}$

$C(x) = 10x^2 + 12x\left(\frac{60}{x^2}\right) = 10x^2 + \frac{720}{x}$

$C'(x) = 20x - \frac{720}{x^2} = \frac{20x^2 - 720}{x^2} = \frac{20(x^3 - 36)}{x^2}$

CP when $x = 0, \pm 36^{1/3}$

Plausible Domain: $(0, 60)$

C': 0 $-$ $36^{1/3}$ $+$ $\sqrt{60}$

C' Δ's from neg to pos at $x = 36^{1/3}$

OR Cand Test:

x	0	$36^{1/3}$	$\sqrt{60}$
$C(x)$	∞	\$327.08	\$692.95

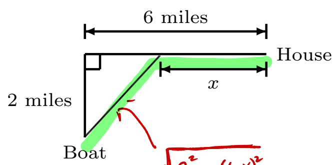
\$327.08 is min cost by Candidates Test

The dimension for the least cost is $36^{1/3}$ feet by $36^{1/3}$ feet by $\frac{60}{36^{2/3}}$

or a square base of 3.302' by 3.302' with height 5.503 feet,

 $(\text{rate})(\text{time}) = \text{distance}$ so $\text{time} = \dfrac{\text{distance}}{\text{rate}}$

14. A person in a rowboat two miles from the nearest point on a straight shoreline wishes to reach a house six miles farther down the shore. If the person can row at a rate of 3 miles per hour and walk at a rate of 5 miles per hour, find the least amount of time required to reach the house. How far from the house should the person land the rowboat?



$\sqrt{2^2+(6-x)^2}$

or $\sqrt{x^2-12x+40}$

Plausible domain walking 6 miles to walking 0 miles.

Primary: $T(x) = \dfrac{\sqrt{x^2-12x+40}}{3} + \dfrac{x}{5}$ hours (min)

Secondary: $\sqrt{x^2-12x+40}$ = dist. rowing

x = dist. walking

$T'(x) = \frac{1}{3}\cdot\frac{1}{2}(x^2-12x+40)^{-1/2}(2x-12) + \frac{1}{5}$

$T'(x) = 0$ when $x = 4.500$ miles

T': $-$ | $+$ at 4.500. T' Δ's from neg to pos

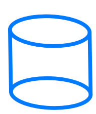
Cand Test

x	0	4.500	6
$T(x)$	2.108	1.733	1.8667

(TI84 $\boxed{\text{Table}}$ Feature helpful here)

The rower should land the boat 4.500 miles from the house to get there in the least time

15. A cylindrical metal container, open at the top, is to have a capacity of 24π cubic inches. The cost of material used for the bottom of the container 15 cents per square inch, and the cost of the material used for the curved part is 5 cents per square inch. Find the dimensions that will minimize the cost of the material, and find the minimum cost.



Primary: $A(r,h) = 15(\pi r^2) + 5(2\pi r h)$ (min)

Secondary: $\text{Vol} = 24\pi = \pi r^2 h$ so $h = \dfrac{24}{r^2}$

$A(r) = 15\pi r^2 + 10\pi r\left(\dfrac{24}{r^2}\right)$

$A(r) = 5\pi\left(3r^2 + \dfrac{48}{r}\right)$

$A'(r) = 5\pi\left(6r - \dfrac{48}{r^2}\right) = \dfrac{5\pi}{r^2}(6)(r^3-8)$

CP when $r = 0, 2$

Plausible Domain is $(0, \sqrt{24})$

A': 0 $-$ 2 $+$ $\sqrt{24}$

A' Δ's from neg to pos at $r=2$, so rel min

If $r=2$, $h = \dfrac{24}{2^2} = 6$

The cheapest can that has a capacity of 24π cubic inches should have a radius of 2 inches and a height of 6 inches.