Worksheet # 15 Series I

1. Convergence Tests

   (a) State the Divergence Test (n^{th} term test). What is the converse and does it hold?

   (b) What is a p-series? For what values of p does it converge and diverge?

   (c) State the Integral Test.

   (d) State the Comparison Test.

   (e) State the Limit Comparison Test.

2. Do each of the following series converge or diverge? For each convergent series find the sum.

$$\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n} \qquad \sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n-1} \qquad \sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^{n-1} \qquad \sum_{n=1}^{\infty}\frac{4^{2n}}{5^{n}}$$

3. For what values of x is the function $f(x)=\sum_{k=1}^{\infty}(x-3)^k$ defined?

4. Write the following decimals as fractions: a) $0.\overline{24}$ b) $3.2\overline{417}$

5. Use the integral test to determine if the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ (b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^3+5}$

6. Determine if the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{10}{n}\right)^{10}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^2+2}}$

   (d) $\displaystyle\sum_{m=1}^{\infty}\frac{m^2+m+1}{3m^2+14m+7}$

   (e) $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n}{2+5^n}$

   (f) $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k^2+4k+3}$

   (g) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n}$

   (h) $\displaystyle\sum_{j=1}^{\infty}\frac{j}{j^2-\cos^2(j)}$

7. Solve the following equation for x: $\displaystyle\sum_{n=2}^{\infty}(1+x)^{-n}=2.$

8. Use facts about geometric series to prove that $.99999999\ldots=1$.

2)

$$\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n \quad \text{converges} \quad \left(\text{geom. series with } |r|=\left|\frac{2}{5}\right|=\frac{2}{5}<1\right)$$

$$=\frac{a}{1-r}=\frac{\frac{2}{5}}{1-\frac{2}{5}}=\frac{\frac{2}{5}}{\frac{3}{5}}=\frac{2}{3}$$

$$\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n-1} \quad \text{diverges} \quad \left(\text{geom. series with } |r|=\left|\frac{5}{2}\right|=\frac{5}{2}>1\right)$$

$$\sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^{n-1} \quad \text{converges} \quad \left(\text{geom. series with } |r|=\left|\frac{1}{3}\right|=\frac{1}{3}<1\right)$$

$$=\frac{a}{1-r}=\frac{\left(\frac{1}{3}\right)^{3-1}}{1-\frac{1}{3}}=\frac{\frac{1}{9}}{\frac{2}{3}}=\frac{1}{9}\cdot\frac{3}{2}=\frac{1}{6}$$

$$\sum_{n=1}^{\infty}\frac{4^{2n}}{5^n}=\sum_{n=1}^{\infty}\frac{16^n}{5^n}=\sum_{n=1}^{\infty}\left(\frac{16}{5}\right)^n \quad \text{diverges}$$

$$\left(\text{geom. series with } |r|=\left|\frac{16}{5}\right|=\frac{16}{5}>1\right)$$

3) $f(x)=\sum_{k=1}^{\infty}(x-3)^k$ is defined as long as the series converges.

$\sum_{k=1}^{\infty}(x-3)^k$ converges if and only if $|x-3|<1$

(because it is a geometric series).

Thus, $f(x)$ is defined for $-1<x-3<1$,

or $2<x<4$.

4) (a) $0.\overline{24} = \sum_{n=1}^{\infty} \frac{24}{100^n} = \frac{a}{1-r} = \frac{\frac{24}{100}}{1-\frac{1}{100}} = \frac{\left(\frac{24}{\cancel{100}}\right)}{\left(\frac{99}{\cancel{100}}\right)} = \frac{24}{99}$

(note: $|r| = \left|\frac{1}{100}\right| < 1$)

(b) $3.2\overline{417} = \frac{32}{10} + \frac{1}{10}\sum_{n=1}^{\infty} \frac{417}{1000^n}$ $\leftarrow \left(\left|\frac{1}{1000}\right| < 1 \right)$

$$= \frac{32}{10} + \frac{1}{10} \cdot \frac{a}{1-r}$$

$$= \frac{32}{10} + \frac{1}{10} \cdot \frac{\left(\frac{417}{1000}\right)}{\left(\frac{999}{1000}\right)}$$

$$= \frac{32}{10} \cdot \frac{1}{10} \cdot \frac{417}{999}$$

$$= \frac{(32)(999) + 417}{(10)(990)}$$

5) (a) $\sum_{n=1}^{\infty} ne^{-n^2}$ converges <u>if and only if</u> $\int_1^{\infty} xe^{-x^2}\,dx$ converges.

First, we compute $\rightarrow \int_1^b xe^{-x^2}\,dx = \int_{u=-1^2}^{u=-b^2} e^u \cdot \frac{1}{(-2)}\,du = -\frac{1}{2} e^u \Big|_{u=-1}^{u=-b^2}$

$u = -x^2$

$\frac{du}{-2} = \frac{\cancel{-2}x\,dx}{\cancel{-2}}$

$$= -\frac{1}{2}\left(e^{-b^2} - e^{-1}\right)$$

Thus, $\int_1^{\infty} xe^{-x^2} = \lim_{b\to\infty} \int_1^b xe^{-x^2} = \lim_{b\to\infty} \left[-\frac{1}{2}\left(e^{-b^2} - e^{-1}\right)\right]$

$$= -\frac{1}{2}\left(\underbrace{\lim_{b\to\infty} \frac{1}{e^{b^2}}}_{=0} - \frac{1}{e}\right) = -\frac{1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e} < \infty$$

Therefore, $\sum_{n=1}^{\infty} ne^{-n^2}$ converges

6) (a) $\boxed{\sum_{n=1}^{\infty} \left(\frac{10}{n}\right)^{10} = 10^{10} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{10}}}$ convergent (p-series, $p=10>1$)

(b) $\boxed{\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}} \leq \sum_{n=1}^{\infty} \frac{n+n}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2n}{n^{\frac{5}{2}}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ convergent

$\boxed{n+1 \leq n+n \text{ comparison test}}$

$\boxed{\text{convergent (p-series, } p=\tfrac{3}{2}>1)}$

(c) $\boxed{\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}} \geq \sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2n^2}} = \sum_{n=1}^{\infty} \frac{2}{\sqrt{3}\cdot n} = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-series, $p=1\leq 1$)

$\boxed{\sqrt{n^2+2} \leq \sqrt{n^2+2n^2} \text{ comparison test}}$

divergent

(d) $\boxed{\sum_{m=1}^{\infty} \frac{m^2+m+1}{3m^2+14m+7}} \longleftrightarrow \sum_{m=1}^{\infty} \frac{1}{3}$

We use Limit comparison:

$$\lim_{m\to\infty} \frac{\left(\frac{m^2+m+1}{3m^2+14m+7}\right)}{\left(\frac{1}{3}\right)} = \lim_{m\to\infty} \frac{3m^2+3m+3}{3m^2+14m+7}$$

dividing numerator and denom. by $3m^2$

$$= \lim_{m\to\infty} \frac{1+\frac{1}{m}+\frac{1}{m^2}}{1+\frac{14}{3m}+\frac{7}{3m^2}} = \frac{1+0+0}{1+0+0} = 1 \quad \boxed{\neq 0 \text{ and finite}}$$

(by limit laws)

Since $\sum_{m=1}^{\infty} \frac{1}{3}$ (obviously) diverges,

so does $\sum_{m=1}^{\infty} \frac{m^2+m+1}{3m^2+14m+7}$, by the Limit Comparison Test

diverges

6) (e) $\boxed{\sum_{n=1}^{\infty} \frac{1+2^n}{2+5^n}} \xleftrightarrow[\text{Test}]{\text{Lim. Comp.}} \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$

We compute $\lim_{n\to\infty} \frac{\left(\frac{1+2^n}{2+5^n}\right)}{\left(\frac{2}{5}\right)^n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n}+\frac{2^n}{2^n}\right)}{\left(\frac{2}{5^n}+\frac{5^n}{5^n}\right)} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n+1}{2\cdot\left(\frac{1}{5}\right)^n+1}$

$= \frac{1}{1} = 1$ (by limit laws)

Thus, since $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ converges (geometric series with $|r| = \left|\frac{2}{5}\right| < 1$),

$\sum_{n=1}^{\infty} \frac{1+2^n}{2+5^n}$ also converges

(f) $\boxed{\sum_{k=1}^{\infty} \frac{2}{k^2+4k+3}} \leq 2\sum_{k=1}^{\infty} \frac{1}{k^2}$ ← converges. (p-series, $p=2>1$)

$\boxed{k^2+4k+3 \geq k^2 \text{ and comparison test}}$

Thus, $\sum_{k=1}^{\infty} \frac{2}{k^2+4k+3}$ also converges

(g) $\boxed{\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$ ← diverges (p-series with $p=1\leq 1$)

$\boxed{e^{\frac{1}{n}} \geq e^0 = 1 \text{ and comparison test}}$

Thus, $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ also diverges

(h) $\boxed{\sum_{j=1}^{\infty} \frac{j}{j^2-\cos^2 j}} \geq \sum_{j=1}^{\infty} \frac{j}{j^2} = \sum_{j=1}^{\infty} \frac{1}{j}$ diverges (p-series with $p=1\leq 1$)

$\cos^2 j \geq 0$

$\Rightarrow j^2-\cos^2 j \leq j^2$

$\Rightarrow \frac{1}{j^2-\cos^2 j} \geq \frac{1}{j^2}$

Thus, the original series also diverges

7)

$$2 = \sum_{n=2}^{\infty} (1+x)^{-n} = \frac{a}{1-r} = \frac{(1+x)^{-2}}{1-(1+x)^{-1}}$$

(because we have a convergent geometric series)

So $2 = \frac{1}{(1+x)^2} \cdot \frac{1}{1-\left(\frac{1}{1+x}\right)} = \frac{1}{(1+x)^2-(1+x)} = \frac{1}{x^2+2x+1-x-1} = \frac{1}{x^2-x}$

or $2(x^2-x) = 1$

or $2x^2-2x-1 = 0$

Thus $x = \frac{-(-2) \pm \sqrt{(-2)^2-4(2)(-1)}}{2(2)} = \frac{2 \pm \sqrt{4+8}}{4}$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$

However, we must have $|1+x| < 1$ in order for the geometric series to converge, so

$\cancel{\frac{1}{2}+\frac{\sqrt{3}}{2}}$ cannot be a solution:

$$\left|1+\frac{1}{2}+\frac{\sqrt{3}}{2}\right| > 1$$

$\frac{1}{2}-\frac{\sqrt{3}}{2}$ is allowed because $\left|1+\frac{1}{2}-\frac{\sqrt{3}}{2}\right| = \frac{|3-\sqrt{3}|}{2} < \frac{2}{2} = 1$

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(8) $0.999\ldots = 0.\overline{9} = \sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{a}{1-r} = \frac{\left(\frac{9}{10}\right)}{1-\left(\frac{1}{10}\right)} = \frac{\cancel{\left(\frac{9}{10}\right)}}{\cancel{\left(\frac{9}{10}\right)}} = 1$

interpretation of the decimal

fact about geometric series