Section 2.2: Derivatives of Polynomials and Exponential Functions

Alternative Notations for the Derivative

If $y = f(x)$, then

$$\frac{dy}{dx} = f'(x)$$

is known as the derivative of y with respect to x.

For example, if we have the function $y = f(x) = 2x^2 - x + 1$, then we can write

$$\frac{dy}{dx} = f'(x) = 4x - 1.$$

For doing intermediate computations, we have the following notation:

$$\frac{d}{dx}(f(x)) = D_x(f(x))$$

Thus, we can say

$$\frac{d}{dx}(2x^2 - x + 1) = D_x(2x^2 - x + 1) = 4x - 1$$

Also, to evaluate a derivative at a point, say $x = a$, we write

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

Hence, if $f'(x) = 4x - 1$, then

$$f'(3) = \left.\frac{dy}{dx}\right|_{x=3} = 4(3) - 1 = 12 - 1 = 11.$$

Basic Derivative Formulas

1. $\frac{d}{dx}(k) = 0$, where k is a constant (for our purposes, a real number).

2. $\frac{d}{dx}(x^k) = k\,x^{k-1}$

3. $\frac{d}{dx}(e^x) = e^x$

4. $\frac{d}{dx}(\sin x) = \cos x$

5. $\frac{d}{dx}(\cos x) = -\sin x$

Example 1: Differentiate $f(x) = 5$.

Solution:

$\blacksquare$

Example 2: Differentiate $y = x^9$.

Solution:

Example 3: Differentiate $y = \sqrt[3]{t}$.

Solution:

Example 4: Differentiate $y = \dfrac{1}{x^3}$.

Solution:

Properties of Differentiation

1. $\dfrac{d}{dx}(k\ f(x)) = k\ f'(x)$ (k is a constant)

2. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Example 5: Differentiate $f(x) = 4x^3 + 5\sin x$.

Solution:

$\blacksquare$

Example 6: Differentiate $y = 6x^3 + 4x^2 - 2x - 2\cos x + 5$.

Solution:

$\blacksquare$

Example 7: Differentiate $f(t) = 3e^t + \sqrt{t} + \dfrac{5}{t^4} - \dfrac{3}{\sqrt[3]{t^2}} - \pi$.

Solution:

$\blacksquare$

Example 8: Differentiate $y = \dfrac{x^2 - 2\sqrt{x}}{x}$.

Solution: Using the following property of fractions that $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$, we first rewrite the function as

$$y = \frac{x^2 - 2\sqrt{x}}{x} = \frac{x^2}{x} - \frac{2x^{1/2}}{x} = x^{2-1} - 2x^{1/2-1} = x - 2x^{-1/2}$$

Differentiating $y = x - 2x^{-1/2}$, we obtain

$$\frac{dy}{dx} = 1 - 2(-\frac{1}{2})x^{-1/2-1}$$

$$\frac{dy}{dx} = 1 + x^{-3/2}$$

$$\boxed{\frac{dy}{dx} = 1 + \frac{1}{x^{3/2}}}$$

$\blacksquare$

Example 9: Find the equation of the line tangent to the graph of $f(t) = \sin t + 2t$ at the point $(\pi, 2\pi)$.

Solution: To find the equation of any line, including a tangent line, we need a point (this is given to be $(\pi, 2\pi)$) and the slope. Recall that the derivative at a point gives the slope of the tangent line at that point. To find a formula for calculating the slope, we calculate the derivative of the function which is given by

$$f'(t) = \cos t + 2$$

Then,

$$\begin{array}{c}\text{Slope of Tangent line} \\ \text{at the point } (\pi, 2\pi) \\ t = \pi\end{array} = m = f'(\pi) = \cos(\pi) + 2 = -1 + 2 = 1$$

Then, using the slope intercept equation of a line (in terms of *t*) given by

$$y = mt + b$$

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we use the slope of the tangent line we just found $m = 1$ to find the equation of the tangent line as follows:

$$\begin{array}{ll} y = (1)t + b & \text{(Substitute the slope } m = 1) \\ y = t + b & \text{(Simplify)} \\ 2\pi = \pi + b & \text{(Use the point } (\pi, 2\pi) \text{ with } t = \pi \text{ when } y = 2\pi) \\ b = 2\pi - \pi = \pi & \text{(Solve for } b) \end{array}$$

Hence, substituting the slope $m = 1$ and $b = \pi$ into $y = mt + b$ gives the tangent line equation:

$$\boxed{y = t + \pi}$$

Graph the function and its tangent line with your calculator

Example 10: Find the point)s on the graph of $y = 8x - 2e^x$ that has a horizontal tangent line.

Solution: On this problem, a horizontal tangent line means that the slope of the tangent line is 0. Since the derivative gives a formula for the slope of the tangent line, we can find the point that gives a tangent line slope of 0 by taking the derivative of the function, setting it equal to 0, and solving for *x*. The result of this calculation is as follows:

$$\frac{dy}{dx} = 8 - 2e^x = 0$$

$$\begin{aligned}
8 - 2e^x &= 0 \\
-2e^x &= -8 && \text{(Subtract 8 from both sides)} \\
e^x &= 4 && \text{(Divide both sides by - 2)} \\
\ln e^x &= \ln(4) && \text{(Take ln of both sides)} \\
x \ \ln e &= \ln(4) && \text{(Use ln property } \ln u^k = k \ln u\text{)} \\
x\,(1) &= \ln(4) && \text{(Recall } \ln e = 1\text{)} \\
x &= \ln(4)
\end{aligned}$$

To complete the problem, we must find the *y*-coordinate of the point by substituting $x = \ln(4)$ back into the original function $y = 8x - 2e^x$. Keeping in mind the inverse property $e^{\ln u} = u$, this is done as follows:

$$y = 8\ln(4) - 2e^{\ln(4)} = 8\ln(4) - (2)(4) = 8\ln(4) - 8$$

Thus, the coordinates of the point that have a horizontal tangent line is

$$\boxed{(\ln(4), 8\ln(4) - 8)}$$

Average Velocity

Suppose the position of a moving object starting from rest is given by the position function $s(t) = 5t^2$ feet where t is the time given in seconds.

t	$s(t) = 5t^2$
0	
1	
2	
3	
4	

We define the average velocity on the time interval from $t = a$ to $t = b$ as follows:

Formula For Average Velocity

$$\begin{array}{c}\text{Average Velocity} \\ \text{on the time interval} \\ [a,b] \\ t = a \ \text{ to } \ t = b\end{array} = \frac{\text{Change in Distance}}{\text{Change in Time}} = \frac{s(b) - s(a)}{b - a}.$$

Example 11: Find the average velocity for the time intervals [1, 3] and [3, 4] for an object if the position starting from rest is given by $s(t) = 5t^2$.

Solution:

$\blacksquare$

Suppose we now desire to find the velocity of the object precisely when $t = 1$ second for the position function $s(t) = 5t^2$.

A method for approximating would involve finding average velocities on an interval that is "close" to $t = 1$.

Example 12: Find the average velocity for the time intervals [1, 1.01] and [1, 1.001] for an object if the position starting from rest is given by $s(t) = 5t^2$.

Solution: To assist in the calculations, we find the position function $s(t) = 5t^2$ at the following times.

$$s(1) = 5(1)^2 = 5(1) = 5$$
$$s(1.01) = 5(1.01)^2 = 5(1.0201) = 5.1005$$
$$s(1.001) = 5(1.001)^2 = 5(1.002001) = 5.010005$$

Then, using the average velocity formula

$$\begin{array}{c}\text{Average velocity on the}\\ \text{time interval}\,[a,b]\\ t=a \text{ to } t=b\end{array} = \frac{\text{Change in distance (height)}}{\text{Change in Time}} = \frac{s(b)-s(a)}{b-a}$$

we obtain

$$\begin{array}{c}\text{Average velocity on the}\\ \text{time interval}\,[1,1.01]\\ t=1 \text{ to } t=1.01\end{array} = \frac{s(1.01)-s(1)}{1.01-1} = \frac{5.1005-5}{0.01} = \frac{0.1005}{0.01} \boxed{= 10.05 \text{ ft/sec}}$$

$$\begin{array}{c}\text{Average velocity on the}\\ \text{time interval}\,[1,1.001]\\ t=1 \text{ to } t=1.001\end{array} = \frac{s(1.001)-s(1)}{1.001-1} = \frac{5.010005-5}{0.001} = \frac{0.010005}{0.001} \boxed{= 10.005 \text{ ft/sec}}$$

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In general, for an object moving from time $t_1 = t$ to time $t_2 = t + h$,

Average Velocity
on the time interval =
$[t, t+h]$

To get the instantaneous velocity at $t_1 = t$, we let $h \to 0$ which gives the following definition.

Instantaneous Velocity and Instantaneous Rate of Change

Given a position function $s(t)$, the instantaneous velocity $v(t)$ is given by the derivative of the position function. That is,

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

In general, if we are given a function $y = f(x)$,

$$\begin{array}{c}\text{Average Rate}\\ \text{of change on}\\ [a, a+h]\end{array} = \frac{f(a+h) - f(a)}{h}$$

$$\begin{array}{c}\text{Instantaneous Rate}\\ \text{of change at}\\ t = a\end{array} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 13: Find the instantaneous velocity for the position function $s(t) = 5t^2$ at $t = 1$.

Solution:

Example 14: The position function representing the height of a freely falling object is given by $s(t) = -16t^2 + v_0 t + s_0$, where v_0 is the initial velocity of the object and s_0 is the initial height of the ball at time $t = 0$. Here the height s is in feet and the time t is in seconds. Suppose someone throws a baseball from 6 feet off the ground with a initial velocity of 100 ft/s.
a. Determine the position and velocity functions for the ball.
b. Find the average velocity for the time intervals [4, 4.1], [4, 4.01], and [4, 4.0001].
c. Find the instantaneous velocity when $t = 4$ and $t = 5$ seconds.
d. Find the time required for the ball to reach ground level.
e. Find the velocity of the coin at impact.

Solution part a: Since the ball starts 6 ft off the ground, the initial height is $s_0 = 6$. The initial velocity is $v_0 = 100$. Substituting into the equation $s(t) = -16t^2 + v_0 t + s_0$ gives the position equation

$$\boxed{s(t) = -16t^2 + 100t + 6}$$

To get the velocity equation, we take the derivative of the position equation $s(t)$. This gives

$$\boxed{v(t) = s'(t) = -32t + 100}$$

Solution part b: Recall that given a time interval $[a, b]$, if s represents the height of the ball (the position) after time t, then

$$\begin{array}{c}\text{Average velocity on the} \\ \text{time interval}\,[a,b] \\ t = a \text{ to } t = b\end{array} = \frac{\text{Change in distance (height)}}{\text{Change in Time}} = \frac{s(b) - s(a)}{b - a}.$$

To find the average velocities for the given intervals, we will use the following calculations:

$s(4) = -16(4)^2 + 100(4) + 6 = -256 + 400 + 6 = 150 \text{ ft}$.
$s(4.1) = -16(4.1)^2 + 100(4.1) + 6 = -268.96 + 410 + 6 = 147.04 \text{ ft}$
$s(4.01) = -16(4.01)^2 + 100(4.01) + 6 = -257.2816 + 401 + 6 = 149.7184 \text{ ft}$
$s(4.0001) = -16(4.0001)^2 + 100(4.0001) + 6 = -256.01280016 + 400.01 + 6 = 149.99719984 \text{ ft}$

Hence,

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$$\begin{array}{c}\text{Average velocity on the} \\ \text{time interval}\,[4, 4.1] \\ t = 4 \text{ to } t = 4.1\end{array} = \frac{s(4.1) - s(4)}{4.1 - 4} = \frac{147.04 - 150}{0.1} = \frac{-2.96}{0.1} = \boxed{-29.6 \text{ ft/sec}}$$

$$\begin{array}{c}\text{Average velocity on the} \\ \text{time interval}\,[4, 4.01] \\ t = 4 \text{ to } t = 4.01\end{array} = \frac{s(4.01) - s(4)}{4.01 - 4} = \frac{149.7184 - 150}{0.01} = \frac{-0.2816}{0.01} = \boxed{-28.16 \text{ ft/sec}}$$

$$\begin{array}{c}\text{Average velocity on the} \\ \text{time interval}\,[4, 4.0001] \\ t = 4 \text{ to } t = 4.0001\end{array} = \frac{s(4.0001) - s(4)}{4.0001 - 4} = \frac{149.99719984 - 150}{0.0001} = \frac{-0.00280016}{0.0001} = \boxed{-28.0016 \text{ ft/sec}}$$

Note that the negative average velocities indicate that the ball is falling down instead of going up.

Solution part c: The average velocities found in part **b** indicate the instantaneous velocity at the specific time $t = 4$ should be close to -28 ft/sec. From part a, we found the equation for the instantaneous velocity at a particular time t to be

$$v(t) = s'(t) = -32t + 100$$

Thus, at $t = 4$ we have

$$\boxed{\begin{array}{c}\text{Instantaneous Velocity} \\ \text{at time } t = 4\end{array} = v(4) = -32(4) + 100 = -128 + 100 = -28 \text{ ft/sec}}$$

We can easily use this same equation to find the velocity at $t = 5$ seconds.

$$\boxed{\begin{array}{c}\text{Instantaneous Velocity} \\ \text{at time } t = 5\end{array} = v(5) = -32(5) + 100 = -160 + 100 = -68 \text{ ft/sec}}$$

Solution part d: When the ball reaches ground level, its height $s = 0$. Thus, the find the time when the ball reaches ground level, we set the position equation

$$s(t) = -16t^2 + 100t + 6 = 0$$

and solve for t. Since this quadratic equation is not easily factorable, we use the quadratic formula to find its approximate solution.

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Recall that the quadratic formula says that the solution to the quadratic equation is given by $at^2 + bt + c = 0$ is given by

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $s(t) = -16t^2 + 100t + 6 = 0$, setting a = -16, b = 100, and c = 6 we obtain

$$t = \frac{-100 \pm \sqrt{(100)^2 - 4(-16)(6)}}{2(-16)}$$

$$t = \frac{-100 \pm \sqrt{10000 + 384}}{-32}$$

$$t = \frac{-100 \pm \sqrt{10384}}{-32}$$

$$t \approx \frac{-100 \pm 101.9}{-32}$$

$$t \approx \frac{-100 - 101.9}{-32}, \frac{-100 + 101.9}{-32}$$

$$t \approx \frac{-201.9}{-32}, \ \frac{1.9}{-32}$$

$$t \approx 6.3, \ \cancel{t \approx -0.05}$$

Thus, the ball hits the ground in approximately $\boxed{\text{6.3 seconds.}}$

Solution part e: From part d, we found out the ball hits the ground after t = 6.3 seconds. To find the velocity when the ball impacts the ground, we substitute t = 6.3 into the velocity equation we found in part a $v(t) = -32t + 100$. Thus,

$$\boxed{\begin{matrix}\text{Velocity when ball} \\ \text{hits ground } (t = 6.3)\end{matrix} = v(6.3) = -32(6.3) + 100 = -101.6 \text{ ft/sec}}$$

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