CALCULUS BC
9.3 – Parametric Equations and Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$,

then the slope of C at the point (x, y) is given by $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\dfrac{dx}{dt} \neq 0$,

and the second derivative is given by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

<u>Ex. 1</u> (Noncalculator)

Given the parametric equations $x = t^2 + 9$ and $y = 4t^5 - 6t^3 + 3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

<u>Ex. 2</u> (Noncalculator)

Given the parametric equations $x = 5\sin t$ and $y = 3\cos t$, write an equation of the tangent line to the curve at the point where $t = \dfrac{2\pi}{3}$.

<u>Ex 3</u> (Noncalculator)

Find all points of horizontal and vertical tangency given the parametric equations

$x = 3t^2 - 6t, \quad y = t^2 - 8t + 9.$

Earlier in the year we learned to find the arc length of a curve C given by $y = h(x)$ over the interval $x_1 \le x \le x_2$ by using the formula $s = \int_{x_1}^{x_2} \sqrt{1 + \left(h'(x)\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$ over the interval $a \le t \le b,$ then

Length of arc for parametric graphs is $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Note that the formula works when the curve does not intersect itself on the interval $a \le t \le b,$ and the curve must be smooth.

<u>Ex. 4</u> (Noncalculator)
Set up an integral expression for the arc length of the curve given by the parametric equations $x = e^{5t+2}$, $y = 3\sin\left(t^2\right)$, $0 \le t \le 4.$ Do not evaluate.

Ex. Which of the following gives the length of the path described by the parametric equations $x=\sin\left(t^3\right)$ and $y=e^{5t}$ from $t=0$ to $t=\pi$?

(A) $\int_0^{\pi}\sqrt{\sin^2\left(t^3\right)+e^{10t}}\,dt$ (B) $\int_0^{\pi}\sqrt{\cos^2\left(t^3\right)+e^{10t}}\,dt$ (C) $\int_0^{\pi}\sqrt{9t^4\cos^2\left(t^3\right)+25e^{10t}}\,dt$

(D) $\int_0^{\pi}\sqrt{3t^2\cos\left(t^3\right)+5e^{5t}}\,dt$ (E) $\int_0^{\pi}\sqrt{\cos^2\left(t^3\right)+e^{10t}}\,dt$

Ex. A curve C is defined by the parametric equations $x=t^2-4t+1$ and $y=t^3$. Which of the following is an equation of the line tangent to the graph of C at the point $(-3, 8)$?

(A) $x=-3$ (B) $x=2$ (C) $y=8$ (D) $y=-\frac{27}{10}(x+3)+8$ (E) $y=12(x+3)+8$

Ex. The length of the path described by the parametric equations $x=\frac{1}{3}t^3$ and $y=\frac{1}{2}t^2$, where $0\le t\le 1$, is given by

(A) $\int_0^1\sqrt{t^2+1}\,dt$ (B) $\int_0^1\sqrt{t^2+t}\,dt$ (C) $\int_0^1\sqrt{t^4+t^2}\,dt$ (D) $\frac{1}{2}\int_0^1\sqrt{4+t^4}\,dt$ (E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2+9}\,dt$

Ex. A particle moves in the xy-plane so that its position for $t \geq 0$ is given by the parametric equations $x = \ln(t+1)$ and $y = kt^2$, where k is a positive constant. The line tangent to the particle's path at the point where $t = 3$ has slope 8. What is the value of k?

Ex. A curve is defined by the parametric equations $x(t) = 3e^{2t}$ and $y(t) = e^{3t} - 1$. What is $\frac{d^2y}{dx^2}$ in terms of t?