CALCULUS BC
Taylor Series and Taylor Polynomials Notes
Lecture 1: Introduction to Power Series

There are a variety of situations in mathematics where it is necessary to express certain functions in other ways:

- At their core, calculators are only capable of basic arithmetic: adding, subtracting, multiplying and dividing. So how does your calculator know what $\sin 1$ is? In fact, your calculator needs a way of expressing $\sin x$ in terms of the four arithmetic operations. Finding a polynomial that approximates the sine function accomplishes that goal.
- We've discussed several functions that do not have antiderivatives; for example, $\cos\left(x^2\right)$ or e^{-x^2}. But what if you desperately need one so that you can calculate (for example) $\int_0^1 e^{-x^2}\,dx$? What if you could approximate e^{-x^2} with a polynomial? Then you could find the antiderivative of each term (easy!) and have a good approximation of the integral!

In both of the above cases, finding a polynomial that approximates a function is critical. In fact, if you allow the polynomial to have an infinite number of terms, then the "infinite polynomial" would no longer be an approximation – it would EQUAL the function!

These "infinite polynomials" (technically, polynomials are defined to be finite, hence the quotation marks) are special examples of **infinite series** that we will focus our attention on in this chapter.

There are four major questions we will answer in this chapter.
We will typically focus on one or two of these at a time:

1. All infinite series either converge or diverge. How can you tell whether a given series converges or diverges?

2. Given any function, how can it be represented as an infinite series (specifically, an "infinite polynomial")?

3. When you express a function as an infinite series, it usually converges for some values of x, but otherwise diverges. How do you find the interval of convergence for a series?

4. Often it's more practical to work with a finite portion of an infinite series, thus obtaining a good approximation of the desired answer. How can you determine how good the approximation is?

An **<u>infinite series</u>** is an expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, which can be written as $\sum_{k=1}^{\infty} a_k$.

The **<u>partial sums</u>** of the series form a sequence:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, \sum_{k=1}^{n} a_k, \ldots$$

If this sequence of partial sums has a limit S as $n \to \infty$, we say that the series **<u>converges</u>** to S .
(We can write: $\sum_{k=1}^{\infty} a_k = S$.)

Otherwise, we say the series **<u>diverges</u>**.

<u>Ex.</u> Given the series $\sum_{n=1}^{\infty} (2n+1) = 3 + 5 + 7 + \ldots$
Find the first <u>five</u> terms of the sequence of partial sums, and list them below.

What do you notice about the results? Do you think that the sequence of partial sums has a limit or a bound?

What about other arithmetic series? Do you think that there are any arithmetic series that converge?

<u>Ex.</u> Given the series $\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \frac{3}{128} + \frac{3}{256} + \frac{3}{512} + \frac{3}{1024} + \ldots$.

Find the first <u>ten</u> terms of the sequence of partial sums, and list them below.

What do you notice about the results? Do you think that the sequence of partial sums has a limit or a bound?

The error in an approximation is the difference between the actual sum of the infinite series and the sum of the terms that you have added. What is the error in your approximation?

$|\text{Error}| = |\text{Actual value} - \text{Approximation}| =$

How could you make the error smaller?

What about other geometric series? Do you think that they all converge?

Given the series $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n} = \frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\ \ldots.$

Find the first <u>five</u> terms of the sequence of partial sums, and list them below:

What do you notice about the results? Do you think that the sequence of partial sums has a limit or a bound?

In Precalculus, you learned a formula to help you find the sum of an infinite geometric series:

$S=\frac{a}{1-r}$ where a is the first term and r is the common ratio

What has to be true for this formula to give you a valid answer?

> The **<u>geometric series</u>**
>
> $$a+ar+ar^{2}+ar^{3}+\ldots+ar^{n-1}+\ldots=\sum_{n=1}^{\infty}ar^{n-1}$$
>
> converges to the sum $\frac{a}{1-r}$ if $|r|<1$, and diverges if $|r|\geq 1$.

The interval $-1<r<1$ is called the **<u>interval of convergence</u>**.

<u>Ex.</u> Tell whether each series converges or diverges. If it converges, give its sum.

(a) $\sum_{n=1}^{\infty}3\left(\frac{1}{2}\right)^{n-1}$

(b) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots$

(c) $\sum_{k=1}^{\infty}\left(\frac{\pi}{e}\right)^{k}$

(d) $\frac{\pi}{2}+\frac{\pi^{2}}{4}+\frac{\pi^{3}}{8}+\ldots$

Now, consider the series $1+x+x^2+x^3+\ldots+x^n+\ldots$

What is a?

What is r?

Therefore, this <u>power series</u> (so called because it has "powers" of x in each term) converges to what function?

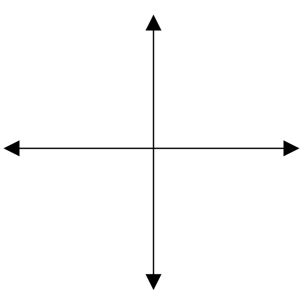
What is the interval of convergence of this series?

Where is the series centered at?

Now let's take a look at some partial sums of this series and see what happens when compared to $f(x)\ldots$

$f(x)=\dfrac{1}{1-x}$

$P_2(x)=1+x+x^2$

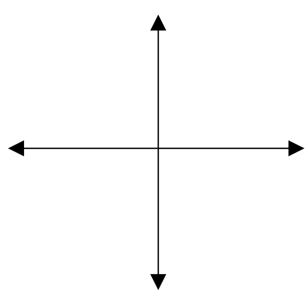


$f\left(\dfrac{1}{2}\right)=$

$P_2\left(\dfrac{1}{2}\right)=$

$f(x)=\dfrac{1}{1-x}$

$P_3(x)=1+x+x^2+x^3$

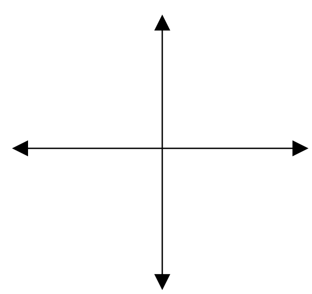


$f\left(\dfrac{1}{2}\right)=$

$P_3\left(\dfrac{1}{2}\right)=$

$f(x)=\dfrac{1}{1-x}$

$P_4(x)=1+x+x^2+x^3+x^3$



$f\left(\dfrac{1}{2}\right)=$

$P_4\left(\dfrac{1}{2}\right)=$

As you can see, by ____, we decrease the error and improve the approximation to $f(x)$!

Write the series we obtained on the previous page here.

Question: How can you manipulate a series for one function to obtain series for other functions?

- You can add or subtract one or more values from a series.

  If you subtract 1 from the original series, you will obtain…

  If you subtract $1+x$ from the original series, you will obtain…

- You can multiply/divide the original series by some factor.

  If you multiply the original series by x, you will obtain…

- You can substitute another expression for *x.*

  If you replace x with $2x$ in the original series, you will obtain…

  If you replace x with $-x$ in the original series, you will obtain…

  Replacing x by $(x-1)$ in the series for $\frac{1}{1+x}$ gives…

Replacing x by x^2 in the series for $\frac{1}{1+x}$ gives…

- <u>You can combine two or more manipulations.</u>

  To find a series for the function $\frac{x}{1-x^3}$, substitute ___________ for x and then multiply the result by __________ …

- <u>You can differentiate the series.</u>

  Taking the derivative of both $\frac{1}{1-x}$ and its series gives…

- <u>You can antidifferentiate the series.</u>

  Finding the antiderivative of both $\frac{1}{1+x}$ and its series gives…

Finding the antiderivative of both $\frac{1}{x}$ and its series gives…

How could we find a series for $\tan^{-1} x$ using a series we've already found?

Challenge: Define a function f by a power series as follows:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$$

(a) Find $f'(x)$

(b) Based on your result in part (a), $f(x)$ must be what function? Cool, huh?

(c) Graph the first three partial sums. What appears to be the interval of convergence?

(d) Graph the next three partial sums. Did you underestimate the interval of convergence?

(e) Write the series for the function ________________.

(f) Write the series for the function ________________.