2.4 The Chain Rule

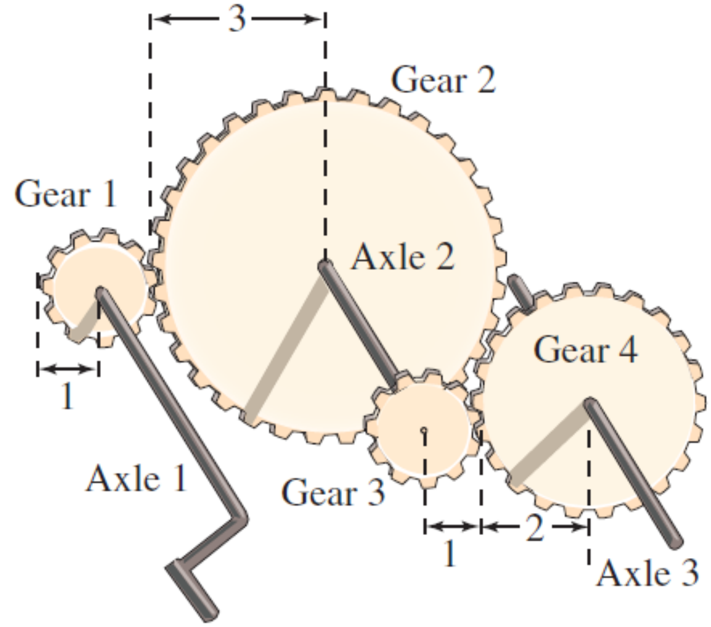
Example 1 – *The Derivative of a Composite Function*

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle.

As the first axle revolves, it drives the second axle, which in turn drives the third axle.

Let *y*, *u*, and *x* represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find *dy*/*du*, *du*/*dx*, and *dy*/*dx* ,*and* show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute

Figure 2.24

Example 1 – *Solution*

Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once.

Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once.

Example 1 – *Solution*

cont'd

So, you can write

$$\frac{dy}{dx} = \begin{bmatrix}\text{Rate of change of first axle} \\ \text{with respect to second axle}\end{bmatrix} \cdot \begin{bmatrix}\text{Rate of change of second axle} \\ \text{with respect to third axle}\end{bmatrix}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3 \cdot 2$$

$$= 6$$

$$= \begin{bmatrix}\text{Rate of change of first axle} \\ \text{with respect to third axle}\end{bmatrix}.$$

In other words, the rate of change of *y* with respect to *x* is the product of the rate of change of *y* with respect to *u* and the rate of change of *u* with respect to *x*.

The Chain Rule

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

The Chain Rule

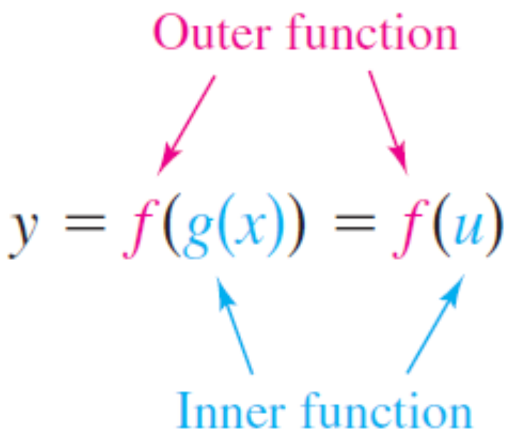
For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

Without the Chain Rule	With the Chain Rule
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if *y* changes *dy*/*du* times as fast as *u*, and *u* changes *du*/*dx* times as fast as *x*, then *y* changes (*dy*/*du*)(*du*/*dx*) times as fast as *x*.

The Chain Rule

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts–an inner part and an outer part.



The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Example 4 – *Applying the General Power Rule*

Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution:

Let $u = 3x - 2x^2$.

Then $f(x) = (3x - 2x^2)^3 = u^3$

and, by the General Power Rule, the derivative is

$$f'(x) = \underbrace{3}_{n}\underbrace{(3x - 2x^2)^2}_{u^{n-1}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{u'}$$ Apply General Power Rule.

$$= 3(3x - 2x^2)^2(3 - 4x).$$ Differentiate $3x - 2x^2$.

Example 7 – *Simplifying by Factoring Out the Least Powers*

Find the derivative of $f(x) = x^2\sqrt{1 - x^2}$.

Solution:

$f(x) = x^2\sqrt{1 - x^2}$	Write original function.
$= x^2(1 - x^2)^{1/2}$	Rewrite.
$f'(x) = x^2 \frac{d}{dx}[(1 - x^2)^{1/2}] + (1 - x^2)^{1/2} \frac{d}{dx}[x^2]$	Product Rule
$= x^2\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(2x)$	General Power Rule
$= -x^3(1 - x^2)^{-1/2} + 2x(1 - x^2)^{1/2}$	Simplify.
$= x(1 - x^2)^{-1/2}[-x^2(1) + 2(1 - x^2)]$	Factor.
$= \dfrac{x(2 - 3x^2)}{\sqrt{1 - x^2}}$	Simplify.

Example 8 – *Simplifying the Derivative of a Quotient*

$$f(x) = \frac{x}{\sqrt[3]{x^2 + 4}}$$ Original function

$$= \frac{x}{(x^2 + 4)^{1/3}}$$ Rewrite.

$$f'(x) = \frac{(x^2 + 4)^{1/3}(1) - x(1/3)(x^2 + 4)^{-2/3}(2x)}{(x^2 + 4)^{2/3}}$$ Quotient Rule

$$= \frac{1}{3}(x^2 + 4)^{-2/3}\left[\frac{3(x^2 + 4) - (2x^2)(1)}{(x^2 + 4)^{2/3}}\right]$$ Factor.

$$= \frac{x^2 + 12}{3(x^2 + 4)^{4/3}}$$ Simplify.

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\, u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\, u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\, u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\, u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\, u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\, u'$$

Example 10 – *The Chain Rule and Trigonometric Functions*

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u} \overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2 \cos 2x$

b. $y = \cos \overbrace{(x - 1)}^{u}$ $\qquad y' = \overbrace{-\sin(x - 1)}^{-(\sin u)} \overbrace{\frac{d}{dx}[x - 1]}^{u'} = -\sin(x - 1)$

c. $y = \tan \overbrace{3x}^{u}$ $\qquad y' = \overbrace{\sec^2 3x}^{(\sec^2 u)} \overbrace{\frac{d}{dx}[3x]}^{u'} = (\sec^2 3x)(3) = 3 \sec^2(3x)$

Trigonometric Functions and the Chain Rule

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules	Let c be a real number, let n be a rational number, let u and v be differentiable functions of x, and let f be a differentiable function of u.		
	Constant Rule: $\frac{d}{dx}[c] = 0$	*(Simple) Power Rule:* $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$	
	Constant Multiple Rule: $\frac{d}{dx}[cu] = cu'$	*Sum or Difference Rule:* $\frac{d}{dx}[u \pm v] = u' \pm v'$	
	Product Rule: $\frac{d}{dx}[uv] = uv' + vu'$	*Quotient Rule:* $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	
	Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)u'$	*General Power Rule:* $\frac{d}{dx}[u^n] = nu^{n-1}u'$	
Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$