3 Applications of Differentiation

3.1 Extrema on an Interval

Objectives

- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

Extrema of a Function

In calculus, much effort is devoted to determining the behavior of a function *f* on an interval *I*.

Does *f* have a maximum value on *I*? Does it have a minimum value? Where is the function increasing? Where is it decreasing?

In this chapter, you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.

Extrema of a Function

Definition of Extrema

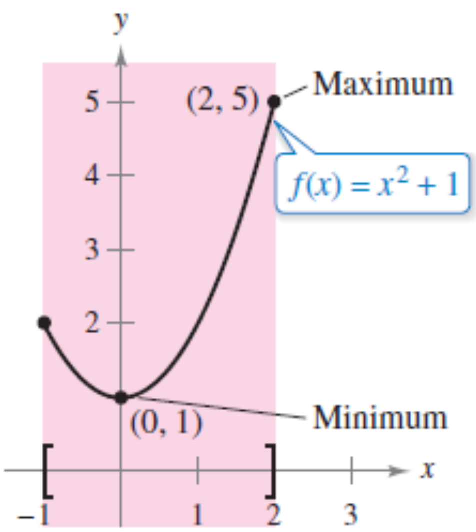
Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval. Extrema can occur at interior points or endpoints of an interval (see Figure 3.1). Extrema that occur at the endpoints are called **endpoint extrema.**

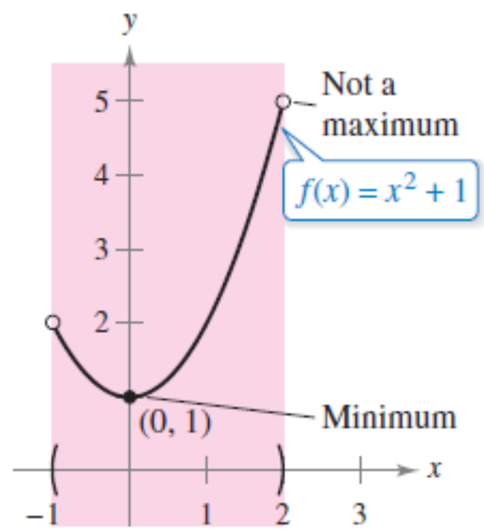
Extrema of a Function

A function need not have a minimum or a maximum on an interval. For instance, in Figures 3.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval [–1, 2] but does not have a maximum on the open interval (–1, 2).



(a) f is continuous, $[-1, 2]$ is closed.

Figure 3.1(a)



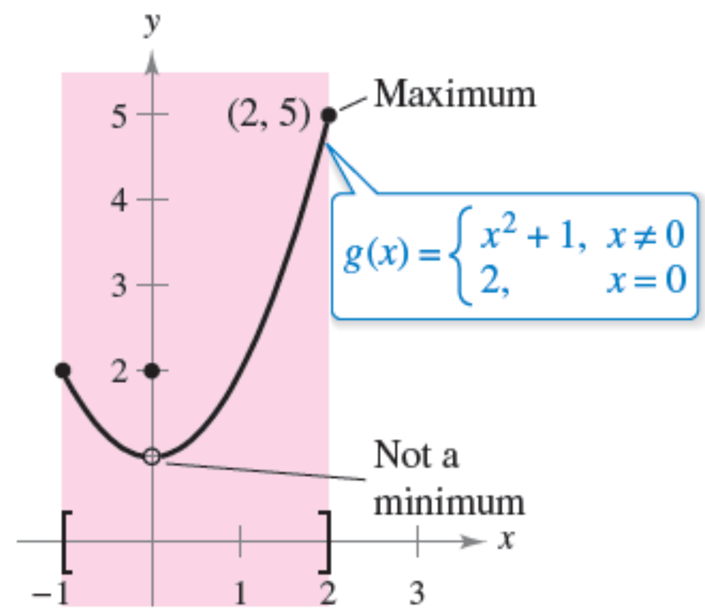
(b) f is continuous, $(-1, 2)$ is open.

Figure 3.1(b)

Extrema of a Function

Moreover, in Figure 3.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval.

This suggests the theorem below.



(c) g is not continuous, $[-1, 2]$ is closed.

Figure 3.1(c)

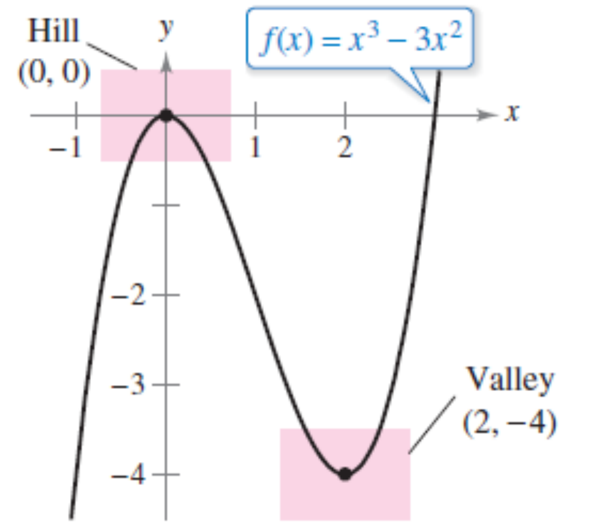
> **THEOREM 3.1 The Extreme Value Theorem**
>
> If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Relative Extrema and Critical Numbers

Relative Extrema and Critical Numbers

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point (0, 0) and a **relative minimum** at the point (2, –4).

Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph.



f has a relative maximum at (0, 0) and a relative minimum at (2, −4).

Figure 3.2

Relative Extrema and Critical Numbers

Such a hill and valley can occur in two ways.

When the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point).

When the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

Example 1 – *The Value of the Derivative at Relative Extrema*

Find the value of the derivative at each relative extremum shown in Figure 3.3.

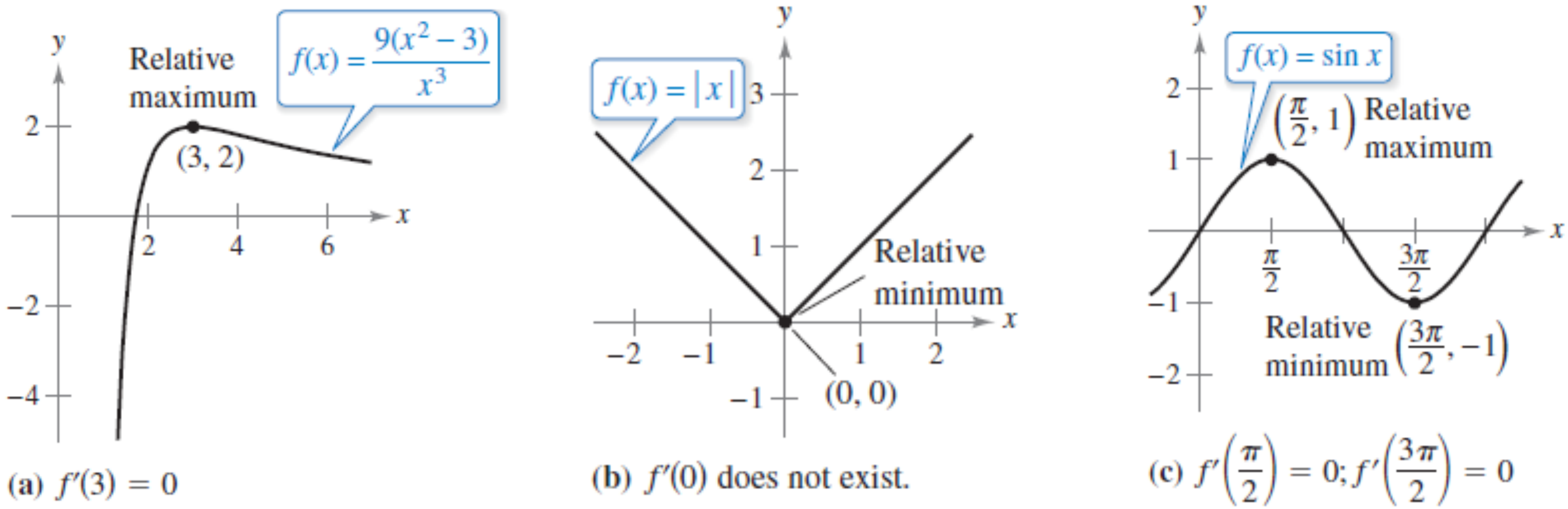


Figure 3.3

Example 1(a) – *Solution*

The derivative of $f(x) = \dfrac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2}$$ Differentiate using Quotient Rule.

$$= \frac{9(9 - x^2)}{x^4}.$$ Simplify.

At the point (3, 2), the value of the derivative is $f'(3) = 0$ [See Figure 3.3(a)].

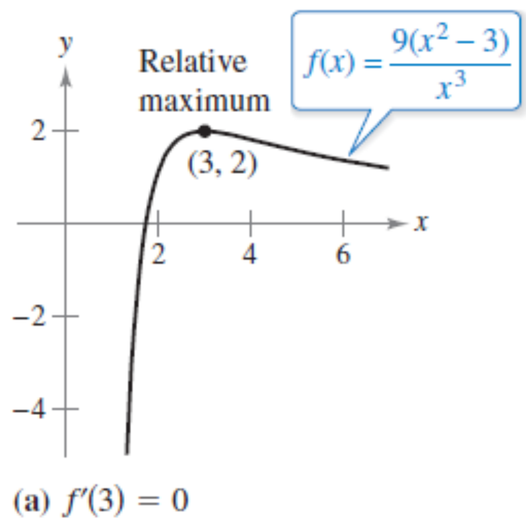
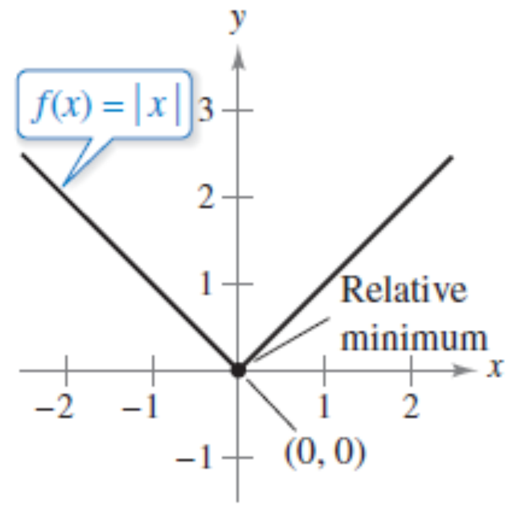


Figure 3.3(a)

Example 1(b) – *Solution*

cont'd

At $x = 0$, the derivative of $f(x) = |x|$ *does not exist* because the following one-sided limits differ [See Figure 3.3(b)].



(b) $f'(0)$ does not exist.

Figure 3.3(b)

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = -1$$

Limit from the left

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

Limit from the right

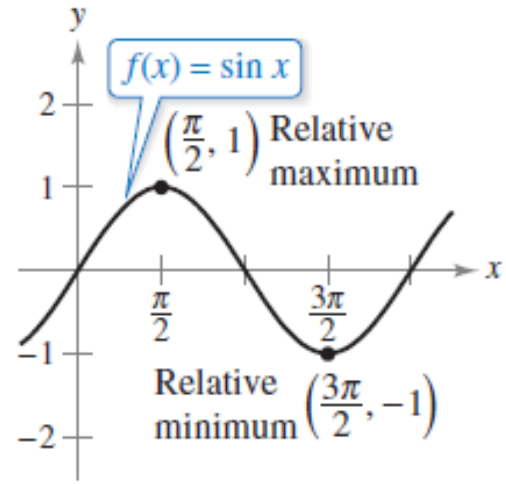
Example 1(c) – *Solution*

cont'd

The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$.

At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ [See Figure 3.3(c)].



(c) $f'\left(\frac{\pi}{2}\right) = 0; f'\left(\frac{3\pi}{2}\right) = 0$

Figure 3.3(c)

Relative Extrema and Critical Numbers

Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The *x*-values at these special points are called **critical numbers.**

Figure 3.4 illustrates the two types of critical numbers.

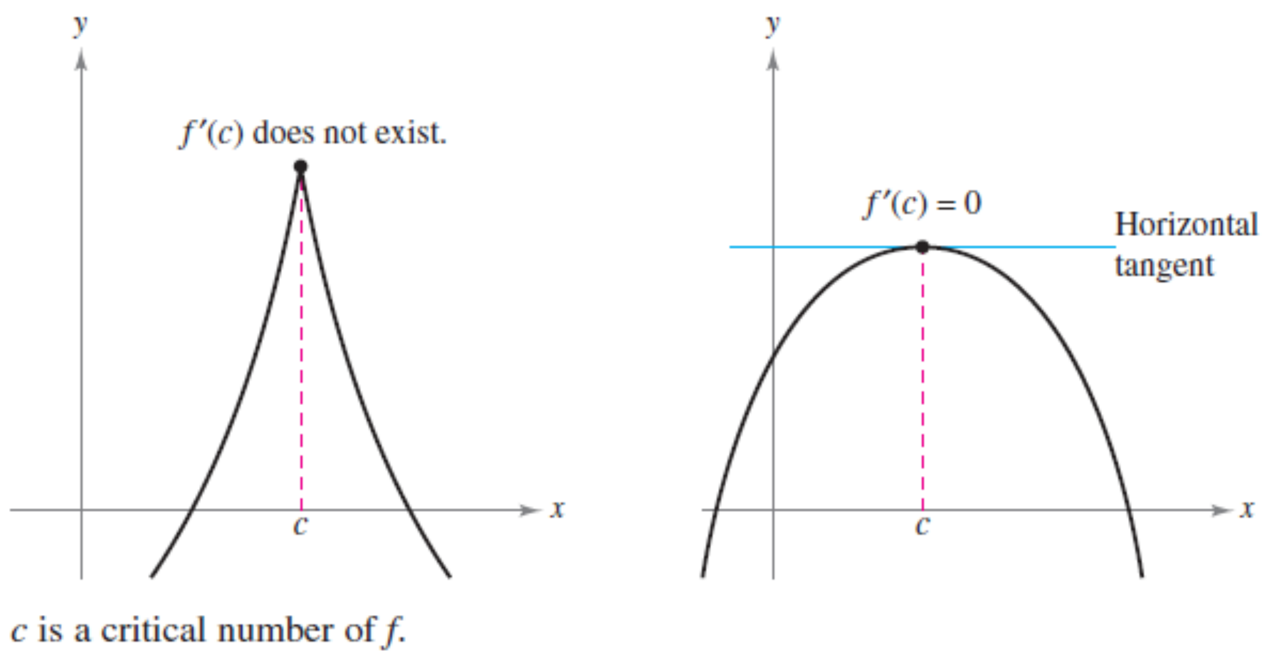


Figure 3.4

Relative Extrema and Critical Numbers

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

Notice in the definition above that the critical number *c* has to be in the domain of *f*, but *c* does not have to be in the domain of *f*′.

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Finding Extrema on a Closed Interval

Finding Extrema on a Closed Interval

Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Example 2 – *Finding Extrema on a Closed Interval*

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval [–1, 2].

Solution:

Begin by differentiating the function.

$f(x) = 3x^4 - 4x^3$ Write original function.

$f'(x) = 12x^3 - 12x^2$ Differentiate.

Example 2 – *Solution*

cont'd

To find the critical numbers of f in the interval $(-1, 2)$, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$

$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1 \qquad \text{Critical numbers}$$

Because f' is defined for all x, you can conclude that these are the only critical numbers of f.

Example 2 – *Solution*

cont'd

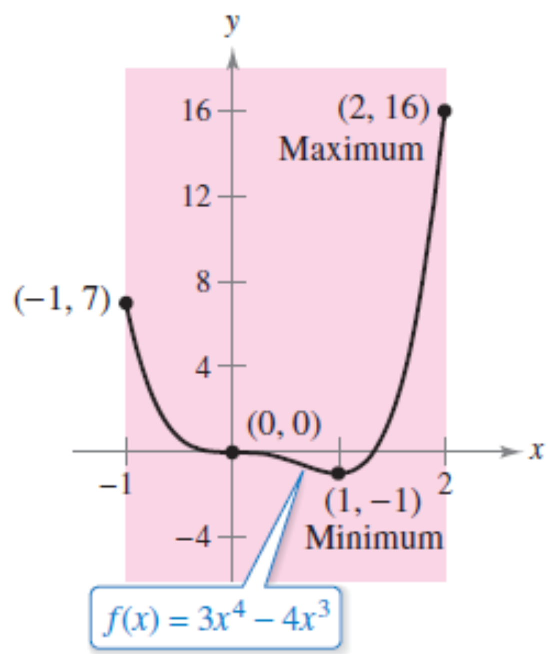
By evaluating *f* at these two critical numbers and at the endpoints of [–1, 2], you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

Example 2 – *Solution*

cont'd

The graph of *f* is shown in Figure 3.5.



On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

Figure 3.5

Example 2 – *Solution*

cont'd

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum.

This tells you that the converse of Theorem 3.2 is not true.

In other words, *the critical numbers of a function need not produce relative extrema*.