Justifications on the AP Exam

The AP Calculus exams take a "4 Pronged" approach. Students are expected to demonstrate their knowledge of calculus concepts in 4 ways.

1. Numerically (Tables/Data)
2. Graphically
3. Analytically (Algebraic equations)
4. Verbally

The verbal component occurs often on the Free Response portion of the exam and requires students to explain and/or justify their answers and work.

It is important that students understand what responses are valid for their explanations and justifications. Sadly, students often understand a concept, but do not earn points on these types of questions because their reasoning is not sufficient. Today, we will take a look at some common errors students make when justifying answer as well as some tips and explanations of what the AP readers require of students.

General Tips and Strategies for Justifications

1. A quality explanation does not need to be too wordy or lengthy. A proper explanation is usually very precise and short. Once you make your statement, STOP WRITING!!! Too often, students give a correct explanation, but continue to further elaborate and end up contradicting themselves or making an incorrect assertion which forfeits any points they could have earned!

2. Students commonly mix ideas in their explanations which cause them to not earn points. For example: "a function f(x) is increasing" is equivalent to writing "f '(x) > 0". However, students often write "f ' (x) is increasing" when they intended to write "f '(x) > 0".

3. Avoid using pronouns in your descriptions. Be specific! Do not write statements that begin with "The function…", "It…", or "The graph…". These are too general and the reader will not assume which function or graph you are referring. Name the functions by starting your statement with the phrase "f(x)…" or "f '(x)…", etc.

4. Know and understand proper mathematical reasons for the ideas covered in this session. Use the precise wording offered today and be assured that these are mathematically correct justifications that will earn you points!

5. Make sure you show that the necessary conditions are met BEFORE you use theorems like the Mean Value Theorem, Intermediate Value Theorem, Continuity, etc…

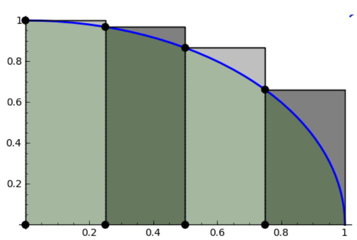
Justifications on the AP Exam

Here are several concepts that have required explanations and justifications on Free Response questions over the past several years. We will address proper justifications of these concepts in this session.

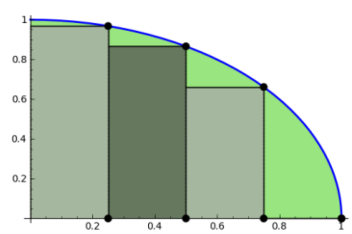
1. Riemann Sums as an over/under approximation of area
2. Relative minimums/maximums of a function
3. Points of inflection on a function
4. Continuity of a function
5. Speed of a particle increasing/decreasing
6. Meaning of a definite integral in context of a problem
7. Absolute minimum/maximum of a function
8. Using Mean Value Theorem
9. Intervals when a function is increasing/decreasing (particle motion)
10. Tangent lines as an over/under approximation to a point on a function

Riemann Sums

If a function is strictly increasing or decreasing on a given interval, then we are able to determine whether right/left Riemann Sums are an over or under approximation of the actual area under the curve.



Left Riemann Sum



Right Riemann Sum

Notice, in the above displays, the function drawn is decreasing. Each rectangle of the Left Riemann Sum is "above" the function but each rectangle of the Right Riemann Sum is "below" the function.

The idea of over/under approximation is fairly intuitive by studying a graph of the function, but stating that the Left Riemann Sum is an over-approximation because each rectangle is above the graph is not acceptable as an explanation. Writing that the rectangles are above the function only restates that it is an over-approximation but does not mathematically explain why we know they must lie above the function. In order to explain WHY a Riemann Sum is an over/under approximation, we need to refer to the behavior of the function. Is the function increasing or decreasing…use this in your reasoning.

Function Behavior	**Left Riemann Sum**	**Right Riemann Sum**
Function is increasing	Under approximation	Over approximation
Function is decreasing	Over approximation	Under approximation

<u>Good Explanation</u>: The left Riemann Sum above is an over-approximation because f(x) is DECREASING on the given interval.

Justifications on the AP Exam

Relative Minimums/Maximums and Points of Inflection

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection.

However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point x = a on the function f(x))	Proper Explanation/Reasoning
Relative Minimum	f(x) has a relative minimum at x = a because f '(x) changes signs from negative to positive when x = a.
Relative Maximum	f(x) has a relative maximum at x = a because f '(x) changes signs from positive to negative when x = a
Point of Inflection	f(x) has a point of inflection at x = a because f "(x) changes sign when x = a.

Continuity

Continuity is an important concept that many calculus students understand intuitively but do not explain well mathematically on the AP exam. The AP exam has included the concept of continuity on the free response portion of the exam each of the past few years.
This concept is usually tested using a piece-wise function on the exam.

Different textbooks may show continuity with different conditions, but essentially a function must meet these three criteria to be continuous at a given point $x = c$:

1. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$
2. $f(c)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

When dealing with piecewise functions, we must take the additional step of checking both the LEFT and the RIGHT sided limits to ensure that the limit does in fact exist.

When showing continuity on the AP exam, students will be expected to show the three conditions above while finding BOTH the left and right sided limits using PROPER form.

Justifications on the AP Exam

Increasing/Decreasing Intervals of a function f(x)

Remember: f '(x) determines whether a function is increasing or decreasing, so always use the sign of f '(x) when determining and justifying whether a function f(x) is increasing or decreasing.

Situation	Explanation
f(x) is increasing on the interval (a,b)	f(x) is increasing on the interval (a,b) because f '(x) > 0
f(x) is decreasing on the interval (a,b)	f(x) is decreasing on the interval (a,b) because f '(x) < 0

Speed Increasing/Decreasing (Particle Motion)

Many students struggle with the concept of speed in particle motion. You are taught that the speed of a particle is the absolute value of velocity.

However, when determining whether the speed of a particle is increasing or decreasing, we cannot just look at the derivative of velocity (acceleration).

In order to determine whether a particle's speed is increasing or decreasing, we must look at BOTH its velocity and acceleration together.

If a particle's velocity and acceleration are in the same direction, then we know its speed will be increasing. In other words, if the velocity and acceleration have the same sign, then its speed is increasing.

On the other hand, if the velocity and acceleration are in opposite directions (different signs), then the speed is decreasing.

When justifying an answer about whether the speed of a particle is increasing/decreasing at a given time, you must find both the velocity and acceleration at that time and make reference to the signs of their values.

Answer	**Possible Justification**
Speed is increasing when t = c	Speed is increasing because v(c) > 0 and a(c) > 0
Speed is increasing when t = c	Speed is increasing because v(c) < 0 and a(c) < 0
Speed is decreasing when t = c	Speed is decreasing because v(c) > 0 and a(c) < 0
Speed is decreasing when t = c	Speed is decreasing because v(c) < 0 and a(c) > 0

Justifications on the AP Exam

IVT and MVT

When working with theorems, a common student mistake is to reference or invoke the conclusion of the theorem without first satisfying the conditions required in the theorem.

To apply the IVT, a function must be continuous on a closed interval. This is often stated in the problem, but you must still acknowledge that the function is indeed continuous.

To apply the MVT, a function must be continuous and differentiable on an open interval. Again, this is often stated in the problem, but you must acknowledge the conditions are met before stating any conclusions.

In recent years, functions that are not differentiable over an open interval have been used to test students' understanding of the MVT. A piecewise function that is not differentiable of an entire open interval are given and students must explain why the MVT is not applicable and thus cannot be used.

Tangent Line Approximations

Unlike a Riemann Sum, determining whether a tangent line is an over/under approximation is not related to whether a function is increasing or decreasing.

When determining (or justifying) whether a tangent line is an over or under approximation, we must use the concavity of the function.

Be careful: We cannot just look at the concavity at the point of tangency. We must look at the concavity on the interval from the point of tangency to the x-value of the approximation!

Example Justification: The approximation of $f(1.1)$ using the tangent line of $f(x)$ at the point $x = 1$ is an over-approximation of the function because $f''(x) < 0$ on the interval $1 < x < 1.1$.

Interpretation of a Definite Integrals

Students have been required to interpret the meaning of a definite integral eight times in the past nine years.

When interpreting the meaning of a definite integral, remember the following:

1. Recognize that a definite integral gives an accumulation or total
2. Always give meaning to the integral in CONTEXT to the problem
3. Give the units of measurement
4. Reference the limits of integration with appropriate units in the context of the problem