CW 2-16

CALCULUS BC
Taylor Series and Taylor Polynomials Notes
Lecture 3: How to Estimate Error for Taylor Polynomials

Let's consider where we are in our study of infinite series. Here's the checklist of goals we have in this chapter.

1. All infinite series either converge or diverge. How can you tell whether a given series converges or diverges? Many tests ...

2. Given any function, how can it be represented as an infinite series (specifically, an "infinite polynomial")? Taylor "sez" $= f(c) + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \dots$

3. When you express a function as an infinite series, it usually converges for some values of x, but otherwise diverges. How do you find the interval of convergence for a series? Ratio Test, check end points

4. Often it's more practical to work with a finite portion of an infinite series, thus obtaining a good approximation of the desired answer. How can you determine how good the approximation is? Lagrange or Alt Ser. Error Bnd

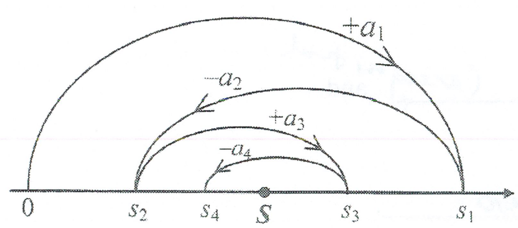
Which ones have we addressed, and to what extent?

In this lecture, we will focus on the fourth bullet point. Let's recall the last lesson, when we used the fifth-degree Taylor polynomial for $\sin x$ to estimate $\sin 1$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin 1 = 1 - \frac{1}{6} + \frac{1}{120}$$

Let's take a look at the partial sums and their errors in relation to the value given by the calculator. Compare the error with the next terms.



Notice that the error in the partial sum is always less than first unused term!

Important Idea: Without a calculator, we can never be sure exactly how well a Taylor polynomial estimates the value of an entire Taylor series. However, we can find an upper bound to the error, so that we can make a statement such as, "The estimate is no more than (upper bound) away from the actual value."

Consider the $\sin 1$ problem again.

$\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} = .841666$

If we extended the Taylor polynomial one more term, what would that next term be?

$\frac{1}{7!}$

OK then, fill in the blanks: The estimate of __.841664__ for $\sin 1$ using the first three non-zero terms of the Taylor polynomial is no more than __$\frac{1}{7!}$__ (the __next__ term) away from the actual value.

How do you know this is true?

What feature(s) of the series for $\sin x$ allows you to make this conclusion this way?

- term are alternating sign
- decreasing in abs
- $\lim_{n\to\infty} a_n = 0$

Alternating Series Remainder (A.S. error bound

If a series has terms that are

(1) __alternating sign__

(2) __decrease in abs. (magnitude.__ and

(3) __has lim of zero__,

then the series converges so that it has a sum S. If the sum S is approximated by the nth partial sum, S_n, then the error in the approximation, $|R_n|$, which equals $|S - S_n|$, will be less than the absolute value of the first omitted or truncated term.

actual estim.

In other words, if the three conditions are met, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will be bounded by the absolute value of the first truncated term.

Ex. Let f be the function defined by $f(x)$ be given by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) First, write the first four nonzero terms and the general term of the Maclaurin series for $\frac{1}{1+x}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$$

(b) Now, write the first four nonzero terms and the general term of the Maclaurin series for $\ln(1+x)$

$$\int_0^x (1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots \, dt$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots + \frac{(-1)^{n+1} t^{n+1}}{n+1} + \dots \Big]_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1} x^{n+1}}{n+1} + \dots$$

(c) Next, write the first four nonzero terms and the general term of the Maclaurin series for $f(x)$.

$$\frac{x}{3} \mapsto x$$

$$x\ln\left(1+\frac{x}{3}\right) = \frac{x^2}{x} - \frac{x^3}{3^2 \cdot 2} + \frac{x^4}{3^3 \cdot 3} + \dots + \frac{(-1)^n x^{n+2}}{3^{n+1} \cdot (n+1)}$$

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Find an upper bound for $|P_4(2) - f(2)|$.

$$|P_4(2) - f(2)| < \left| \frac{-2^5}{3^4 \cdot 4} \right|$$

Ex. The Taylor series about $x = 2$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given $f^{(n)}(2) = \frac{(-1)^n}{3^n}$ and $f(2) = \frac{1}{3}$.

(a) Write the second-degree Taylor polynomial for f about $x = 2$.

$f'(2) = \frac{-1}{3}$

$f''(2) = \frac{1}{9}$

$f'''(2) = \frac{-1}{27}$

$\frac{1}{3} - \frac{1}{3}(x-2) + \frac{1}{9}\frac{(x-2)^2}{2}$

(b) Show that the second-degree Taylor polynomial for f about $x = 2$ approximates $f(3)$ with an error less than $\frac{1}{100}$.

$\left| P_2(2) - f(2) \right| < \left| \frac{-1}{27 \cdot 3!}(3-2)^3 \right| = \frac{1}{162} < \frac{1}{100}$

Ex. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^{n+1}(x-n)^n}{2^n \cdot n} + \cdots$

(b) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$f(1.2) = 1 - \frac{1}{2}(1.2 - 1) + \frac{1}{8}(1.2 - 1)^2$

(c) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$\left| f(1.2) - P_2(1.2) \right| < \frac{1}{24}(1.2-1)^3 \Big| < \frac{1}{1{,}000}$

What happens if a series is not alternating? What then? How can you find an upper bound for the error in that situation?

Answer: Very similarly! However, rather than use the actual next term of the series, we start with a formula that mimics the next term. First, though, some further explanation of what error is…

Given: $f(x) =$ power series in x

A **partial sum** is the sum of the first "few" terms of the series.
The **tail** is the rest of the terms of the series after a partial sum.
the **remainder** is the number you get by "adding" all the terms in the tail.

So $f(x) =$ partial sum + remainder, or another way to say this is $f(x) = P_n(x) + R_n(x)$

The **error** is the error you make by assuming $f(x) =$ the partial sum.

So the **error** is the same number as the **remainder** (obvious, but subtle)
An **error bound** is a number known to be greater than the absolute value of the remainder.

Now, consider what Monsieur Lagrange is credited with showing. The **LAGRANGE REMAINDER** (the error) is exactly equal to the first term of the tail, but with its derivative evaluated not at $x = c$ (about which the series is expanded) but at some number z which is between c and the value of x at which you are evaluating the function. As this value of z comes from (repeated) application of the Mean Value Theorem, there is often no way of knowing exactly what z equals. But if you can find a number that is an upper bound for the derivative between c and x, then you can find a **LAGRANGE ERROR BOUND**.

Taylor's Theorem

If $f(x)$ is expanded as a Taylor series about $x = c$ and x is a number in the interval of convergence, then there is a number z between a and x such that the remainder, R_n, after the partial sum S_n, is given by the **Lagrange form**

$$R_n = \frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}$$

If M is the maximum value of $\left|f^{n+1}(x)\right|$ on the interval between c and x, then the **Lagrange error bound** is

$$\left|R_n\right| \le \frac{M}{(n+1)!}\left|x-c\right|^{n+1}$$

When applying Taylor's Theorem, you should not expect to be able to find the exact value of z. (If you could do this, an approximation would not be necessary.) Rather, you are trying to find bounds for $f^{(n+1)}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is. This bound is called **Lagrange's form of the remainder** or the **Lagrange error bound**.

3 ways:
1) $|f^{(+)}(z)| \le x$
2) sin or cos ≤ 1
3) graph

Ex. 1 The function f has derivatives of all orders for all real numbers x. Assume that

$f(2)=6,\ f'(2)=4,\ f''(2)=-7,\ f'''(2)=8.$

(a) Write the third-degree Taylor polynomial for f about $x=2$, and use it to approximate $f(2.3)$. Give three decimal places.

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$

$= 6 + 4(x-2) - \frac{7}{2}(x-2)^2 + \frac{8}{3!}(x-2^3$

$f(2.3) \approx P_3(2.3) = 6.921$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 9$ for all x in the closed interval [2, 2.3]. Use this information to find a bound for the error in the approximation of $f(2.3)$ found in part (a).

$M=9$

$R_3(2.3) = \left|f(2.3) - P_3(2.3)\right| \le \frac{9}{4!}(2.3-2)^4 = .0030375$

(c) Use your answers to parts (a) and (b) to find an interval [a, b] such that $a \le f(2.3) \le b$. Give three decimal places.

$6.9179625 \le f(2.3) \le 6.9240375$

(d) Could $f(2.3)$ equal 6.922? Explain why or why not.

yes on interval described in part(c)

(e) Could $f(2.3)$ equal 6.927? Explain why or why not.

No, outside the interval

Ex. 2 Let f be the function given by $f(x) = \sin\left(5x + \dfrac{\pi}{3}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

$f'(x) = 5\cos\left(5x + \frac{\pi}{3}\right)$

$f''(x) = -25\sin\left(5x + \frac{\pi}{3}\right)$

$f'''(x) = -125\cos\left(5x + \pi/3\right)$

$f^{(4)}(x) = 625\sin\left(5x + \frac{\pi}{3}\right)$

$\leq 625(1)$

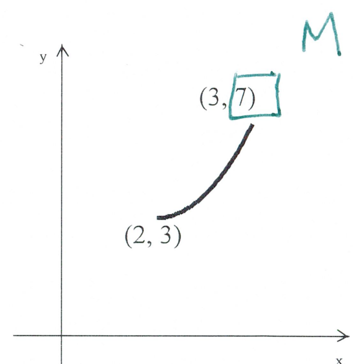
M. for sin

$$P_3(x) = \frac{\sqrt{3}}{2} + \frac{5}{2}x - \frac{25\sqrt{3}}{4}x^2 - \frac{125}{12}x^3$$

(b) Use the Lagrange error bound to show that $\left|f\left(\dfrac{1}{15}\right) - P\left(\dfrac{1}{15}\right)\right| < \dfrac{1}{1200}$.

$$\left|f\left(\tfrac{1}{15}\right) - P_3\left(\tfrac{1}{15}\right)\right| < \left|\frac{625}{4!\,15^4}\right| = \frac{1}{1944} < \frac{1}{1200}$$

Ex. 2 Let f be a function that has derivatives of all orders. Assume $f(2) = 4$, $f'(2) = -\dfrac{1}{3}$, $f''(2) = -\dfrac{1}{5}$, $f'''(2) = \dfrac{3}{7}$, and the graph of $f^{(4)}(x)$ on [2, 3] is shown on the right. The graph of $y = f^{(4)}(x)$ is increasing on [2, 3].



Graph of $y = f^{(4)}(x)$

(a) Find the third-degree Taylor polynomial $P(x)$ about $x = 2$ for the function f.

$$P_3(x) = 4 - \frac{1}{3}(x-2) - \frac{1}{10}(x-2)^2 - \frac{1}{14}(x-2)^3$$

(b) Use your answer to part (a) to estimate the value of $f(2.8)$.

$$4 - \frac{.8}{3} - \frac{.8^2}{10} - \frac{.8^3}{14}$$

(c) Use information from the graph of $y = f^{(4)}(x)$ to show that $|f(2.8) - P(2.8)| < \dfrac{1}{8}$.

$$|f(2.8) - P_3(2.8)| \leq \frac{7}{4!}(.8)^4 = .119466 < .125$$