Chapter Four: Integration

4.1 Antiderivatives and Indefinite Integration

Definition of Antiderivative – A function F is an antiderivative of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

Representation of Antiderivatives – If *F* is an antiderivative of *f* on an interval *I*, then *G* is an antiderivative of *f* on the interval *I* if and only if *G* is of the form $G(x) = F(x) + C$, for all *x* in *I* where *C* is a constant.

This tells us that all antiderivatives of a function differ only by a constant C.

Examples: Find an antiderivative and then find the general antiderivative.

1. $y = 3$

A possible anti-der is $Y = 3x$. In general, the anti-der is $Y = 3x + C$

2. $f(x) = 2x$

A possible $F(x)$ is $F(x) = x^2$. In general, $F(x) = x^2 + C$

3. $f(x) = 5x^4$

A possible antiderivative is $F(x) = x^5 + 7$. In general, $F(x) = x^5 + C$.

Since any constant has derivative 0, we use C for an arbitrary constant in the antiderivative

Notation: If we take the differential form of a derivative, $\frac{dy}{dx} = f(x)$, and rewrite it in the form $dy = f(x)dx$ we can find the antiderivative of both sides using the integration symbol $\int$. That is,

$$y = \int dy = \int f(x)dx = F(x) + C$$

Each piece of this equation has a name that I will refer to: The integrand is *f(x)*, the variable of integration is given by *dx*, the antiderivative of *f(x)* is *F(x)*, and the constant of integration is *C*. The term indefinite integral is a synonym for antiderivative.

Note: Differentiation and anti-differentiation are "inverse" operations of each other. That is, if you find the antiderivative of a function *f*, then take the derivative, you will end up back at *f*. Similarly, if you take the derivative, the antiderivative takes you back.

Some Basic Integration Rules:

$$\int 0dx = C \qquad \int kdx = kx + C \qquad \int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

* important distinction!

We can also consider all the trig derivatives and go backwards to find their integrals.

Examples: For each function, rewrite then integrate and finally simplify.

1. $\int \sqrt[3]{x}dx = \int x^{1/3}dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4}x^{4/3} + C$

   $\sqrt[3]{x} = x^{1/3}$

2. $\int \frac{1}{4x^2}dx = \int \frac{1}{4}x^{-2}dx = \frac{1}{4}\int x^{-2}dx = \frac{1}{4}\frac{x^{-2+1}}{-2+1} + C = \frac{1}{4}\frac{x^{-1}}{-1} + C = \frac{x^{-1}}{-4} + C$

   $= -\frac{1}{4x} + C$

3. $\int \frac{1}{x\sqrt{x}}dx = \int x^{-3/2}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = \frac{x^{-1/2}}{-\frac{1}{2}} + C = -2x^{-1/2} + C = \frac{-2}{\sqrt{x}} + C$

   simplify $\boxed{x\sqrt{x} = x \cdot x^{1/2} = x^{1+1/2} = x^{3/2}}$

4. $\int x(x^3+1)dx = \int (x^4 + x)dx = \frac{x^{4+1}}{4+1} + \frac{x^{1+1}}{1+1} + C = \frac{x^5}{5} + \frac{x^2}{2} + C$

   No product rule for integrals!

5. $\int \frac{1}{(3x)^2}dx = \int \frac{1}{9x^2}dx = \frac{1}{9}\int x^{-2}dx = \frac{1}{9}\frac{x^{-2+1}}{-2+1} + C = \frac{1}{9}\frac{x^{-1}}{-1} + C = \frac{-1}{9x} + C$

On the homework: 6. $\int \frac{1}{x\sqrt[5]{x}}dx = \int x^{-6/5}dx = \frac{x^{-6/5+1}}{-\frac{6}{5}+1} + C = \frac{x^{-1/5}}{-\frac{1}{5}} + C = -5x^{-1/5} + C = \frac{-5}{\sqrt[5]{x}} + C$

WebAssign makes it look weird, but this is $x \cdot \sqrt[5]{x}$ which is $x \cdot x^{1/5} = x^{6/5}$

Examples: Find the indefinite integral and check the result by differentiation.

1. $\int (12-x)\,dx = 12x - \frac{x^2}{2} + C$

Check: $\frac{d}{dx}\left(12x - \frac{x^2}{2} + c\right) = 12 - \frac{2x}{2} + 0 = 12 - x$ ✓

2. $\int (8x^3 - 9x^2 + 4)\,dx = \frac{8x^4}{4} - \frac{9x^3}{3} + 4x + C = 2x^4 - 3x^3 + 4x + C$

Check: $\frac{d}{dx}(2x^4 - 3x^3 + 4x + c) = 8x^3 - 9x^2 + 4 + 0 = 8x^3 - 9x^2 + 4$ ✓

3. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{1}{2}\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

Rewrite $\sqrt{x} + \frac{1}{2\sqrt{x}} = x^{1/2} + \frac{1}{2}x^{-1/2}$

$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{1}{2}\frac{x^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + c\right) = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) + \frac{1}{2}x^{-1/2} + 0 = \sqrt{x} + \frac{1}{2\sqrt{x}}$ ✓

4. $\int \frac{x^2+2x-3}{x^4}dx = \int(x^{-2}+2x^{-3}-3x^{-4})\,dx = \frac{x^{-1}}{-1}+2\frac{x^{-2}}{-2}-\frac{3x^{-3}}{-3}+C = -x^{-1}-x^{-2}+x^{-3}+C$

$= -\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}+C$

No quotient rule for integrals!

$\frac{x^2+2x-3}{x^4} = \frac{x^2}{x^4}+\frac{2x}{x^4}-\frac{3}{x^4} = x^{-2}+2x^{-3}-3x^{-4}$

Check: $\frac{d}{dx}\left(-x^{-1}-x^{-2}+x^{-3}+C\right) = -(-1x^{-2})-(-2x^{-3})+(-3x^{-4})+0 = x^{-2}+2x^{-3}-3x^{-4}$ ✓

5. $\int (2t^2-1)^2\,dt = \int(4t^4-4t^2+1)\,dt = 4\frac{t^5}{5}-4\frac{t^3}{3}+t+C = \frac{4}{5}t^5-\frac{4}{3}t^3+t+C$

Rewrite $(2t^2-1)^2 = 4t^4-4t^2+1$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5-\frac{4}{3}t^3+t+C\right) = \frac{4}{5}(5t^4)-\frac{4}{3}(3t^2)+1+0 = 4t^4-4t^2+1$ ✓

6. $\int (t^2-\cos t)\,dt = \int t^2\,dt-\int \cos t\,dt = \frac{t^3}{3}-\sin t+C$

Check: $\frac{d}{dt}\left(\frac{t^3}{3}-\sin t+C\right) = \frac{1}{3}(3t^2)-\cos t+0 = t^2-\cos t$ ✓

7. $\int (\theta^2+\sec^2\theta)\,d\theta = \frac{\theta^3}{3}+\tan\theta+C$

↑ what has derivative equal to $\sec^2\theta$?

Check: $\frac{d}{d\theta}\left(\frac{\theta^3}{3}+\tan\theta+C\right) = \frac{3\theta^2}{3}+\sec^2\theta+0$

$= \theta^2+\sec^2\theta$ ✓

8. $\int \sec y(\tan y-\sec y)\,dy = \int(\sec y\tan y-\sec^2 y)\,dy = \sec y-\tan y+C$

Still no product rule!

Check: $\frac{d}{dy}(\sec y-\tan y+C) = \sec y\tan y-\sec^2 y+0$

$= \sec y\tan y-\sec^2 y$ ✓

Example: Find the equation of y given $\frac{dy}{dx} = 2x - 1$ that has the particular point (1, 1) as part of its solution set.

$$y = \int dy = \int (2x-1)\,dx = 2\frac{x^2}{2} - x + C = x^2 - x + C$$

Since (1,1) is a solution, we can substitute to find a specific value of C:

$1 = (1)^2 - (1) + C$

$1 = 1 - 1 + C$

$1 = C$

Solution: $y = x^2 - x + 1$

Example: Solve the differential equation.

1. $f'(x) = 6x^2, f(0) = -1$

$$f(x) = \int 6x^2\,dx = 6\frac{x^3}{3} + C = 2x^3 + C$$

$-1 = 2(0)^3 + C$

$-1 = C$

$f(x) = 2x^3 - 1$

2. $f'(p) = 10p - 12p^3, f(3) = 2$

$$f(p) = \int (10p - 12p^3)\,dp = \frac{10p^2}{2} - \frac{12p^4}{4} + C = 5p^2 - 3p^4 + C$$

$2 = 5(3)^2 - 3(3)^4 + C$

$2 = 45 - 243 + C$

$200 = C$

$f(p) = 5p^2 - 3p^4 + 200$

3. $h''(x) = \sin x, h'(0) = 1, h(0) = 6$

$$h'(x) = \int h''(x)\,dx = \int \sin x\,dx = -\cos x + C$$

$1 = -\cos(0) + C$

$1 = -1 + C$

$2 = C$

$h'(x) = -\cos x + 2$

$$h(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C$$

$6 = -\sin(0) + 2(0) + C$

$6 = C$

$\boxed{h(x) = -\sin x + 2x + 6}$

Example: A particle, initially at rest, moves along the *x*-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

a) Find the velocity and position functions for the particle.

b) Find the values of *t* for which the particle is at rest.

$a(t) = \cos t$

$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C$

at rest |
so $C = 0$

part a → $v(t) = \sin t$

$x(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C$

$3 = -\cos(0) + C$
$3 = -1 + C$
$4 = C$

part a → $x(t) = -\cos t + 4$

b) particle at rest ⇒ velocity is 0

$v(t) = \sin t = 0$ when $t = k\pi$ for any integer k