Related Rates:
Falling ladders, Inflating Balloons, Lengthening Shadows, Draining Tanks.

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Another use of the Chain Rule:

find the rates of change of two or more related variables that are changing with respect to *time*.

The textbook's 4 Guidelines......

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

but word problems need some translation into our "math-y" way of solving for unknowns....

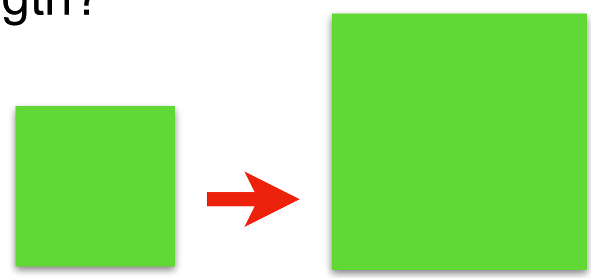
The table below lists examples of mathematical models involving rates of change.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\frac{dx}{dt} = 50$ mi/h when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\frac{dV}{dt} = 10\ \mathrm{m}^3/\mathrm{h}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min
A population of bacteria is increasing at a rate of 2000 per hour.	$x =$ number in population $\frac{dx}{dt} = 2000$ bacteria per hour

5 Steps to Related Rates:

- 1: What are you given? (think of units)
- 2. What are you asked for? (think of units)
- 3: What equation(s) connect these?
- 4: Implicitly Differentiate (d/dt)
- 5: Evaluate @ known value (think of units)

The sides of a square are increasing at a rate of 5 cm/sec.
How fast is the area increasing when the sides measure 15 cm in length?



$$x = \text{side length}$$

$$\frac{dx}{dt} = 5 \text{ cm/sec}$$

$$\text{A} = \text{Area}$$

$$\left.\frac{dA}{dt}\right|_{x=15} = \text{? cm}^2/\text{sec}$$

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$$A = x^2$$

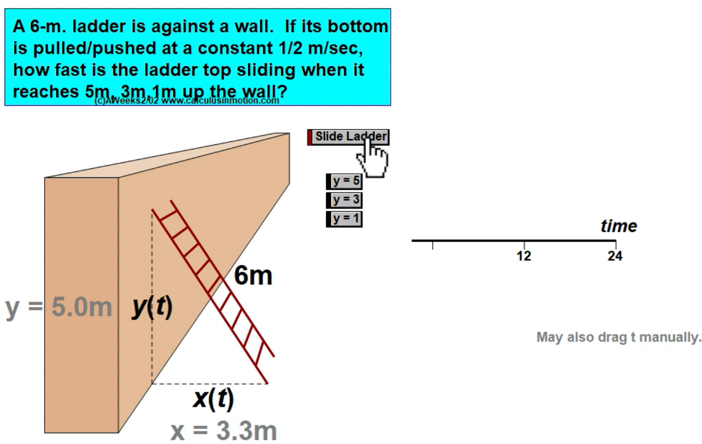
$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dA}{dt} = 2(15)(5) \text{ cm}^2/\text{sec}$$

The area is increasing at rate of 150 square centimeters per second when the side reaches 15 cm in length

Falling Ladders

A 6 meter ladder is against a wall. The top end of the ladder is sliding down the wall as the bottom is being moved. When the top end is 5 meters from the ground it is being pulled away from the wall at a rate of 1/2 meter per second. How fast is the ladder top sliding when the top of the ladder is 5 meters from the ground?



- 1: What are you given? (think of units)
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$$\frac{dx}{dt} = \frac{1}{2} \text{ meters per second}$$

$$\left.\frac{dy}{dt}\right|_{y=5} = \ ? \text{ m/sec}$$

$$x^2 + y^2 = 6^2$$

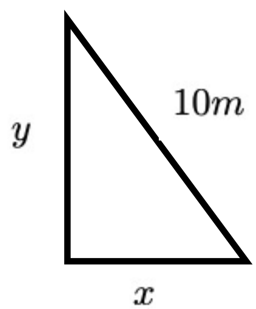
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(\sqrt{6^2 - 5^2})(\tfrac{1}{2}) + 2(5)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{\sqrt{11}}{10} \text{ meters per second}$$

The ladder is sliding down the wall at a rate of about 0.332 meters per second.

A 10 meter ladder is against a wall. The top end of the ladder is sliding down the wall. When the top end is 6 meters from the ground it is sliding down at 2 meters per second. How fast is the bottom moving away from the wall at this moment?



- 1: What are you given? (think of units)
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$$\left.\frac{dy}{dt}\right|_{y=6m} = -2 \text{ meters/sec}$$

$$\left.\frac{dx}{dt}\right|_{y=6m(\text{ so } x=8m)} = \; ? \text{ meters/sec}$$

$$x^2 + y^2 = 10^2$$

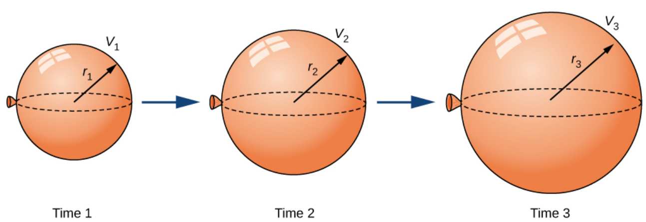
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(8)\frac{dx}{dt} + 2(6)(-2) = 0$$

$$\frac{dx}{dt} = \frac{24}{16} \text{ meters/sec}$$

The bottom of the ladder is moving away from the wall at a rate of 1.5 meters per second when the height of the ladder is 6 meters from the ground

A spherical Balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface Area of the Balloon at the moment when the surface area is 64π square centimeters.



$$r = \text{radius of balloon}$$

$$V = \text{volume} = \tfrac{4}{3}\pi r^3$$
$$A = \text{surface area} = 4\pi r^2$$

$$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$$

$$\left.\frac{dA}{dt}\right|_{A=64\pi cm^2(\text{ so } r=4cm \ \ A=64\pi cm^2)} = \ ? \text{ meters/sec}$$

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$$\frac{dV}{dt} = \frac{4\pi}{3}(3r^2)\cdot\frac{dr}{dt}$$

$$10 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=4} = \frac{10}{64\pi}$$

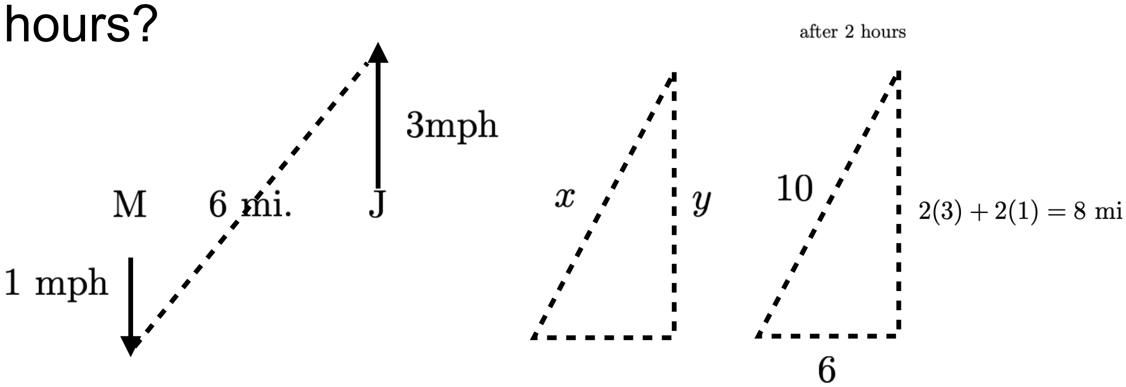
$$\frac{dA}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=4} = 32\pi \cdot \frac{10}{64\pi}$$

$$\left.\frac{dA}{dt}\right|_{r=4} = 5\text{cm}^2/\text{sec}$$

The surface area of the balloon is increasing at the rate of 5 square centimeters per second when the surface area of the balloon is 64π square centimeters

Joe is standing 6 miles due east of Moe. If Joe walks due North at 3 mph while Moe walks due south at 1 mph, at what rate is the distance between them changing after 2 hours?



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$x =$ distance between

$y =$ vertical distance

$6 =$ horizontal distance

$\frac{dy}{dt} = 3 + 1 = 4$ mph

$$\left.\frac{dx}{dt}\right|_{2hours} = \ ?$$

$$6^2 + y^2 = x^2$$

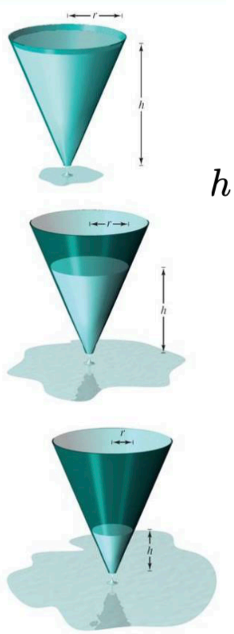
$$0 + 2y \cdot \frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

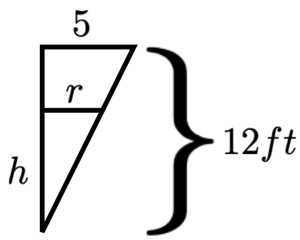
$$2(8)(4) = 2(10) \cdot \left.\frac{dx}{dt}\right|_{x=10}$$

$$\left.\frac{dx}{dt}\right|_{x=10} = 3.2 \text{ mph}$$

After 2 hours, the distance between Joe and Moe is increasing at a rate of 3.2 mph

Water is drained out of a full conical tank that measures 10 feet across the top and 12 feet deep. If the water is draining out at the rate of 10 cubic feet per minute, what is the rate of change of the depth of water when the depth is 8 feet?





$r =$ radius of top of water

$h =$ height of water

$$V = \frac{\pi}{3} r^2 h \text{ and } \frac{dV}{dt} = -10 \text{ ft}^3/\text{min}$$

$$\left.\frac{dh}{dt}\right|_{h=8} = \text{? ft/min}$$

$$\frac{r}{5} = \frac{h}{12}$$

$$r = \frac{5h}{12}$$

$$r^2 = \frac{25h^2}{144}$$

$$V = \frac{\pi}{3}\left(\frac{25h^2}{144}\right) h = \frac{25\pi}{432} h^3$$

$$\frac{dV}{dt} = \frac{25\pi}{432} 3h^2 \cdot \frac{dh}{dt}$$

$$-10 = \frac{25\pi}{432} 3 \cdot 8^2 \cdot \left.\frac{dh}{dt}\right|_{h=8}$$

$$\left.\frac{dh}{dt}\right|_{h=8} = -\frac{9}{10\pi} \text{ft/min}$$

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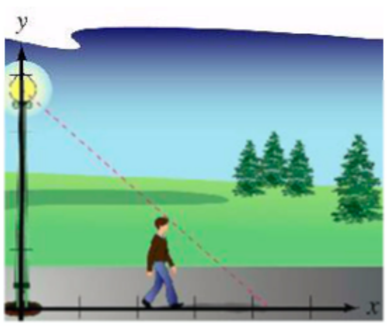
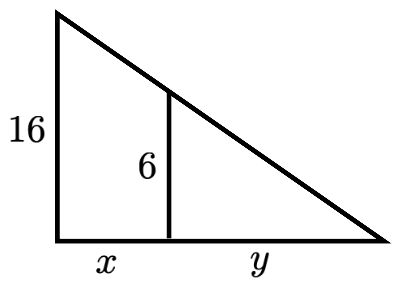
The water is draining at a rate of 9/10π (about 0.286) feet per minute (3.438 inches/minute) when the water depth is at 8 feet.

A 6 feet tall man walks away from a 16 foot street light at a rate of 2 feet per second. As he walks, the shadow lengthens. When he is 8 feet from the streetlight, what is the rate at which the shadow is lengthening?





$$\frac{x+y}{16} = \frac{y}{6}$$

$$6x + 6y = 16y$$

$x =$ man's distance from streetlight

$$\left.\frac{dx}{dt}\right|_{x=8} = 2 \text{ ft/sec.}$$

$y =$ length of shadow

$$\left.\frac{dy}{dt}\right|_{x=8} = \ ?$$

$$6x = 10y$$

$$6\frac{dx}{dt} = 10\frac{dy}{dt}$$

$$6(2) = 10\frac{dy}{dt}$$

$$\frac{dy}{dt} = 1.2 \text{ feet / sec}$$

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The length of the shadow is growing at a rate of 1.2 feet per second.