The Harmonic Divergence
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Has anyone *not* struggled with the mystery of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$, whose terms diminish to zero but yet diverges?[1]

Jakob Bernoulli did, around 1700. Struck by the unexpected result, Jakob wrote in his *Tractatus* (quoted in Bill Dunham's *Journey Through Genius*):

> As the finite encloses an infinite series
> And in the unlimited limits appear,
> So the soul of immensity dwells in minutia
> And in the narrowest limits no limits inhere.
> What joy to discern the minute in infinity!
> The vast to perceive in the small, what divinity .

Old news. It's not exactly news today, but the harmonic series diverges.

It *was* news once. The first known proof is attributed to Nicole Oresme (1323–1382), a Parisian scholar and (later) Bishop of Lisieux, a good 300 years before calculus itself existed. Oresme's proof still appears in modern calculus texts: First group the terms like this—

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

so that successive groups have 1, 1, 2, 4, 8, . . . terms, and so on forever. Since the sum of each group is at least 1/2, the entire series must diverge.

A proof by grouping. In the 1500's an even simpler and cleverer proof appeared, based on grouping terms into *threes*: If H denotes the entire series, then

$$H = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \dots$$

Next, a little thought shows that the *average* of each triple of terms exceeds the *middle* term. Thus the first group exceeds 1, the second 1/2, the third 1/3, etc. That is,

$$H = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \dots > 1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots = 1 + H.$$

In other words, $H > 1 + H$—not likely for any *finite* quantity H.

[1] From *Wikipedia*: The Harmonic Convergence was a loosely organized new age spiritual event which occurred on August 16 and August 17, 1987, when groups of people gathered in various sacred sites and "mystical" places all over the world to usher in a new era, a date based primarily on the Maya calendar, but also on interpretations of European and Asian astrology.

The Harmonic Convergence was supposed to be a global awakening to love and unity through divine transformation. It was initiated in 1987 by Jose Arguelles. According to his interpretation of Maya cosmology (an interpretation held as completely unfounded by Mayanist scholarship), this date was the end of twenty-two cycles of 52 years each, or 1144 years in all. . . . According to Arguelles and others, the Harmonic Convergence also began the final 26-year countdown to the end of the Mayan Long Count in 2012, which would be the "end of history" and the beginning of a new 5,125-year cycle. All the evils of the modern world – war, materialism, violence, abuses, injustice, governmental abusive power, etc. – would end with the birth of the 6th Sun and the 5th Earth on December 21, 2012.

Problem 1. Give another proof of divergence, this time based on grouping terms in *pairs*.

Integral tests. Yet another road to harmonic divergence involves the integral test. Here's a telegraphic version:

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \int_{x=1}^{\infty} \frac{1}{x}\, dx = \lim_{t\to\infty} \ln t = \infty.$$

That's easy enough—but a little inelegantly "disproportionate" because improper integrals are in some sense more sophisticated objects than infinite series.

A better use of integrals, arguably, is in estimating partial sums of the harmonic series. One useful inequality "traps" the nth partial sum H_n (aka the nth *harmonic number*) between two integrals:

$$\int_1^{n+1} \frac{1}{x}\, dx < H_n = \sum_{k=1}^{n} \frac{1}{k} < 1 + \int_1^{n} \frac{1}{x}\, dx.$$

Problem 2. Give a "picture proof" of the preceding inequality, and deduce the more interesting inequality

$$\ln(n+1) < H_n < 1 + \ln n,$$

What does the inequality say for $n = 10^{10}$? For $n = 10^{100}$? (Note: Can you make reasonable estimates without *any* technology? For reference, see the footnote.[2])

Exploring the harmonic numbers numerically

How do the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ behave numerically? Here are some *Maple* samples:

Some partial sums of the harmonic series

n grows linearly				n grows exponentially			
n	H_n	n	H_n	n	H_n	n	H_n
10	2.92897	200	3.59774	2	1.50000	2048	8.20208
20	3.59774	300	3.99499	4	2.08333	4096	8.89510
30	3.99499	400	4.27854	8	2.71786	8192	9.58819
40	4.27854	500	4.49921	16	3.38073	16384	10.2813
50	4.49921	600	4.67987	32	4.05850	32768	10.9744
60	4.67987	700	4.83284	64	4.74389	65536	11.6676
70	4.83284			128	5.43315		
80	4.96548			256	6.12434		
90	5.08257			512	6.81652		
100	5.18738			1024	7.50918		

[2] $\ln(10) \approx 2.303$

Problem 3. Compare the left and right blocks of the table above—pretend that you don't already know whether the series converges or diverges.

(a) What, if anything, does each block suggest to a naive reader?

(b) Mind the gaps $H_{2n} - H_n$ in the right-hand block. What well-known function behaves in a similar way?

(c) Observe that $H_{200} - H_{100} = \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \cdots + \frac{1}{200}$. Interpret the sum as an approximating sum for an appropriate integral to explain why the value is around $\ln 2$. Does the sum under- or overestimate the integral?

An integral formula. Euler gave the analytic formula $H_n = \int_0^1 \frac{1 - x^n}{1 - x}\, dx$ for the harmonic numbers.

Problem 3. Prove Euler's integral formula.

The Euler–Mascheroni constant

Euler showed around 1735 that the sequence $H_n - \ln n$ converges. The limiting value is the Euler–Mascheroni constant, $\gamma \approx 0.5772$. Mascheroni calculated at least 19 correct digits around 1790; by 1812, Gauss and others knew at least 40 digits.

Problem 4. Show that the sequence $H_n - \ln n$ is (a) positive; and (b) decreasing, and therefore converges to a limit.

Alternating harmonic series

It's well known that the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$

converges *conditionally*, but not absolutely, and that the limit is $\ln 2 \approx 0.6931$. This follows (with $x = 1$) from the *Mercator series* identity

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} + \ldots,$$

which holds for $-1 < x \leq 1$. (Mercator was a contemporary of Newton.)

Problem 5. Derive the Mercator series identity by integrating (from 0 to x) the finite geometric series identity

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-t)^{n-1} + \frac{(-t)^n}{1+t}.$$

Do any convergence issues arise for $-1 < x \leq 1$?

Rearranging the alternating harmonic series. The alternating harmonic series (AHS) can be rearranged to converge to *any* desired limit L. This is true abstractly, of course, for *every* conditionally convergent series (remember why?).

Because of its special form, the AHS behaves well with respect to some natural and simple rearrangements. For example, we might take the first r positive summands, then the first s negative summands, then the next r positive summands, then the next s negative summands, and so on. With $r = 1$ and $s = 2$, we get the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots .$$

With, say, *Mathematica* we can calculate partial sums numerically, and thus estimate the limit. In this case the limiting value turns out to be $\dfrac{\ln 2}{2}$.

It turns out that for *any* pair (r, s) of positive integers, the AHS rearranged in successive blocks of r positive and s negative terms — we'll call this rearrange(r, s) — has limit $\dfrac{\ln(4r/s)}{2}$. The following problems explore this fact.

Problem 6. Following is a sketch of how you might convince someone that rearrange$(2, 3)$ has limit $\dfrac{\ln(8/3)}{2}$. Let

$$\begin{aligned} f(x) &= x^3 + \frac{x^9}{3} - \frac{x^4}{2} - \frac{x^8}{4} - \frac{x^{12}}{6} + \frac{x^{15}}{5} + \frac{x^{21}}{7} - \frac{x^{16}}{8} - \frac{x^{20}}{10} - \frac{x^{24}}{12} + \cdots \\ &= \frac{1}{2}\left[\ln(1 + x^3) - \ln(1 - x^3) + \ln(1 - x^4)\right] \\ &= \frac{1}{2}\ln\left(\frac{(1 + x^3)(1 - x^4)}{1 - x^3}\right). \end{aligned}$$

Now find $\displaystyle\lim_{x \to 1^-} f(x)$.

Problem 7. In order to find the general formula for the limit of rearrange(r, s), you want to assign powers of x to the summands so that:

(i) If you separate the positive terms from the negative terms, the exponents for the positive terms form an arithmetic sequence, and the exponents for the negative terms form an arithmetic sequence.

(ii) The r positive summands and the s negative negative summands in the kth block all have exponents that are both larger than the exponents of the terms in the $k - 1$st block and smaller than the exponents of the terms in the $k + 1$st block.

Figure out how to do this.

Solution to Problem 1. If the series converges to H, then

$$H = \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\cdots > \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\ldots$$
$$= 1+\frac{1}{2}+\frac{1}{3}+\ldots = H.$$

Thus we've shown $H > H$, which is clearly absurd, so the series must diverge.

Solution to Problem 2. With $n = 10^{10}$ we get $23.03 < H_n < 24.03$. With $n = 10^{100}$ we get $230.3 < H_n < 231.3$. For reference, *Maple* gives $H_{10^{100}} \approx 230.83$.

Solution to Problem 3. Expand the numerator.

Solution to Problem 4. Part (a) is immediate by comparing an integral and an approximating sum. For (b) the key inequality is

$$\frac{1}{n+1} \le \ln(n+1) - \ln(n) \le \frac{1}{n},$$

which follows from a look the integral $\displaystyle\int_n^{n+1} \frac{1}{x}\,dx$.

Solution to Problem 5. Integrating both sides gives

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt.$$

The last summand tends to zero as $n \to \infty$. If $0 \le x \le 1$ we have

$$\left|\int_0^x \frac{(-t)^n}{1+t}\,dt\right| < \int_0^x t^n\,dt < \int_0^1 t^n\,dt = \frac{1}{n+1}.$$

Solution to Problem 6. Write out the power series s_1, s_2, and s_3 for $\ln(1+x^3)$, $\ln(1-x^3)$, and $\ln(1-x^4)$, respectively. Then note that the power series $\dfrac{s_1 - s_2 + s_3}{2}$ boils down to the function $f(x)$ in Problem 6.

Solution to Problem 7. One approach is to mimic Problem 6: Write out the power series s_1, s_2, and s_3 for $\ln(1+x^a)$, $\ln(1-x^a)$, and $\ln(1-x^b)$, respectively, and then choose a and b so that condition (ii) of Problem 7 holds. This turns out to happen when $\dfrac{a}{b} = \dfrac{s}{2r}$, or $a = s$ and $b = 2r$. The analogue of $f(x)$ in Problem 6 then becomes

$$f(x) = \frac{1}{2}\ln\left(\frac{(1+x^a)(1-x^a)}{1-x^b}\right). = \frac{1}{2}\ln\left(\frac{(1+x^s)(1-x^{2r})}{1-x^s}\right).$$

Taking the limit as $x \to 1^-$ finishes the job.

Another approach uses the formula

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + \gamma + E(n), \quad \text{where} \quad \lim_{n\to\infty} E(n) = 0.$$

Now we have

$$\sum_{k=1}^{n} \left[\left(\frac{1}{2(k-1)r+1} + \frac{1}{2(k-1)r+3} + \cdots \frac{1}{2kr-1} \right) - \left(\frac{1}{2(k-1)s+2} + \frac{1}{2(k-1)s+4} + \cdots + \frac{1}{2ks} \right) \right]$$

$$\begin{aligned}
&= \left(1 + \frac{1}{3} + \cdots + \frac{1}{2nr-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2ns}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2nr}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{nr}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{ns}\right) \\
&= \ln(2nr) + \gamma + E(2nr) - \frac{1}{2}\big(\ln(nr) + \gamma + E(nr) + \ln(ns) + \gamma + E(ns)\big) \\
&= \frac{1}{2}\big(\ln(4n^2r^2) - \ln(nr) - \ln(ns)\big) + E(2nr) - \frac{1}{2}E(nr) - \frac{1}{2}E(ns) \\
&= \frac{1}{2}\ln(4r/s) + E(2nr) - \frac{1}{2}E(nr) - \frac{1}{2}E(ns).
\end{aligned}$$

As n approaches infinity, this approaches $\frac{\ln(4r/s)}{2}$.