The Harmonic Series Diverges Again and Again*

Steven J. Kifowit
Prairie State College

Terra A. Stamps
Prairie State College

The harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

is one of the most celebrated infinite series of mathematics. As a counterexample, few series more clearly illustrate that the convergence of terms to zero is not sufficient to guarantee the convergence of a series. As a known series, only a handful are used as often in comparisons.

From a pedagogical point of view, the harmonic series provides the instructor with a wealth of opportunities. The leaning tower of lire (Johnson 1955) (a.k.a the book stacking problem) is an interesting hands-on activity that is sure to surprise students. Applications such as Gabriel's wedding cake (Fleron 1999) and Euler's proof of the divergence of $\sum 1/p$ (p prime) (Dunham 1999, pages 70–74) can lead to some very nice discussions. And the proofs of divergence are as entertaining as they are educational.

A quick survey of modern calculus textbooks reveals that there are two very popular proofs of the divergence of the harmonic series: those fashioned after the early proof of Nicole Oresme and those comparing $\sum_{k=1}^{n} 1/k$ and $\int_1^{n+1} 1/x\,dx$. While these proofs are notable for their cleverness and simplicity, there are a number of other proofs that are equally simple and insightful. In this article, the authors survey some of these divergence proofs. Throughout, H_n is used to denote the nth partial sum of the harmonic series. That is,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n = 1, 2, 3, \ldots.$$

A common thread connecting the proofs is their accessibility to first-year calculus students.

*To appear in *The AMATYC Review*, Spring 2006

The Proofs

Though the proofs are presented in no particular order, it seems fitting to begin with the classical proof of Oresme.

Proof 1

Nicole Oresme's proof dates back to about 1350. While the proof seems to have disappeared until after the Middle Ages, it has certainly made up for lost time.

PROOF: Consider the subsequence $\{H_{2^k}\}_{k=0}^{\infty}$.

$$\begin{aligned}
H_1 &= 1 = 1 + 0\left(\frac{1}{2}\right), \\
H_2 &= 1 + \frac{1}{2} = 1 + 1\left(\frac{1}{2}\right), \\
H_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right), \\
H_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right).
\end{aligned}$$

In general,

$$H_{2^k} \ \geq \ 1 + k\left(\frac{1}{2}\right).$$

Since the subsequence $\{H_{2^k}\}$ is unbounded, the sequence $\{H_n\}$ diverges. ■

Using the same type of argument, one can show that for any positive integer M,

$$H_{M^k} \ \geq \ 1 + k\left(\frac{M-1}{M}\right).$$

A slight variation on this theme is presented next.

Proof 2

The following proof is derived from one given by Honsberger (1976, page 98).

PROOF: There are 9 one-digit numbers, 1 to 9, whose reciprocals are greater than 1/10. Therefore

$$H_9 > \frac{9}{10}.$$

There are 90 two-digit numbers, 10 to 99, whose reciprocals are greater than 1/100. Therefore

$$H_{99} > \frac{9}{10} + \frac{90}{100} = 2\left(\frac{9}{10}\right).$$

Continuing with this reasoning, it follows that

$$H_{10^k-1} > k\left(\frac{9}{10}\right).$$

Since the subsequence $\{H_{10^k-1}\}$ is unbounded, the sequence $\{H_n\}$ diverges. ■

Proof 3

Credit for this proof goes to Pietro Mengoli. His proof dates back to the middle of the 17th century. The presentation given here is similar to Dunham's (1990, pages 204–205).

PROOF: First notice that

$$\frac{1}{n-1} + \frac{1}{n+1} = \frac{2n}{n^2-1} > \frac{2n}{n^2} = \frac{2}{n}, \quad n = 2, 3, 4, \ldots$$

so that

$$\frac{1}{2} + \frac{1}{4} > \frac{2}{3}, \qquad \frac{1}{5} + \frac{1}{7} > \frac{2}{6}, \qquad \frac{1}{8} + \frac{1}{10} > \frac{2}{9}, \quad \text{etc.}$$

Now suppose that the harmonic series converges with sum S. Then

$$\begin{aligned} S &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \cdots \\ &> 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \cdots \\ &= 1 + S. \end{aligned}$$

The contradiction $S > 1 + S$ concludes the proof. ■

The inequality used by Mengoli,

$$\frac{1}{n-1} + \frac{1}{n+1} > \frac{2}{n}, \quad n = 2, 3, 4, \ldots,$$

is a special case of the harmonic mean/arithmetic mean inequality:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} < \bar{x}.$$

Based on this inequality, one can also show that for positive integers k and j,

$$\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{k+j} \;>\; \frac{2j+2}{j+2k}.$$

From this, a number of other proofs can be derived. For example, consecutive terms of the harmonic series could be grouped in such a way that the sum of each group is at least one. This would lead to the subsequence $\{H_{(3^n-1)/2}\}$ whose nth term is bounded below by n:

$$H_{(3^n-1)/2} \;\geq\; n, \qquad n = 1, 2, 3, \ldots .$$

Proof 4

This proof makes use of an interesting example of a bounded, monotone sequence. It is well known that the sequence under investigation converges to $\ln 2$, but its limit is not relevant to the proof.
PROOF: Consider the sequence $\{S_n\}_{n=1}^{\infty}$, where

$$S_n = H_{2n} - H_n = \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

Since

$$S_{n+1} - S_n = \frac{1}{2n+1} - \frac{1}{2n+2} > 0,$$

the sequence $\{S_n\}$ is increasing. Also,

$$\frac{1}{2} = S_1 \leq S_n = \sum_{k=n+1}^{2n} \frac{1}{k} \leq \sum_{k=n+1}^{2n} \frac{1}{n+1} = \frac{n}{n+1} < 1,$$

and therefore the sequence $\{S_n\}$ is positive and bounded above. It follows that $\{S_n\}$ converges to a positive number between $1/2$ and 1. Since $S_n = H_{2n} - H_n$, the sequence $\{H_n\}$ must diverge. ■

Notice that the fact that $S_n \geq 1/2$ is enough to show that the sequence of partial sums $\{H_n\}$ is not a Cauchy sequence and is therefore divergent.

Proof 5

Honsberger (1976, page 178) gives this proof as a solution of one of his exercises. In addition to a familiar exponent law, the proof makes use of the inequality

$e^x > 1 + x$, which holds for any nonzero x.

PROOF: Consider the sequence $\{e^{H_n}\}_{n=1}^{\infty}$.

$$\begin{aligned}
e^{H_n} &= \exp\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\
&= e^1 \cdot e^{1/2} \cdot e^{1/3} \cdot e^{1/4} \cdots e^{1/n} \\
&> (1+1) \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{4}\right) \cdots \left(1 + \frac{1}{n}\right) \\
&= \left(\frac{2}{1}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{4}{3}\right) \cdot \left(\frac{5}{4}\right) \cdots \left(\frac{n+1}{n}\right) \\
&= n + 1.
\end{aligned}$$

Since $\{e^{H_n}\}$ is unbounded, $\{H_n\}$ is unbounded. ■

Proof 6

The next proof also came from Honsberger (1976, page 102), but credit was given to Leonard Gillman.

PROOF: Suppose that the harmonic series converges with sum S. Then

$$\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= S.
\end{aligned}$$

The contradiction $S > S$ concludes the proof. ■

Notice that a whole family of similar proofs can be derived by considering terms in groups of three, four, etc.

Proof 7

The following proof was given by Cusumano (1998).

PROOF: Suppose that the harmonic series converges with sum S. Then

$$\begin{aligned} S &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \\ &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\left(\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{12}\right)+\left(\frac{1}{3}+\frac{1}{30}\right)+\left(\frac{1}{4}+\frac{1}{56}\right)+\cdots. \end{aligned}$$

It follows that

$$S = S + \left(\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\cdots\right).$$

This contradiction concludes the proof. ■

This proof is very closely related to the previous proof. In fact, the terms on opposite sides of the inequality of Proof 6 differ by 1/2, 1/12, 1/30, 1/56, etc. Just as Gillman's proof has variations, which are based on grouping larger collections of terms, so there are variations on Cusumano's. For example, after placing terms into groups of three, one would find that

$$S = S + \left(\frac{5}{6}+\frac{14}{120}+\frac{23}{504}+\frac{32}{1320}+\cdots\right).$$

Proof 8

This next proof was first presented by Cohen and Knight (1979) and later by Ecker (1997).

PROOF: Suppose the harmonic series converges with sum S. Then

$$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}+\cdots = \frac{1}{2}S.$$

Therefore the sum of the odd-numbered terms,

$$1+\frac{1}{3}+\cdots+\frac{1}{2n-1}+\cdots,$$

must be the other half of S. However this is impossible since

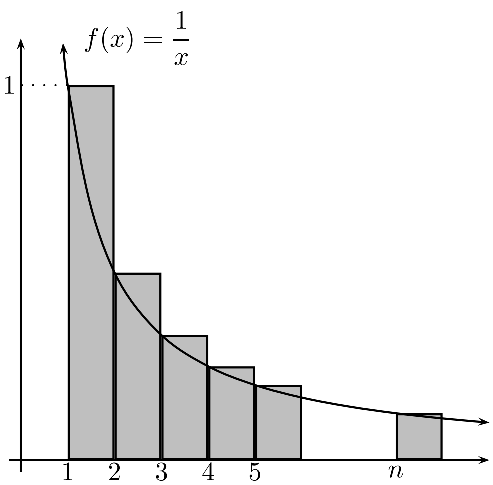
$$\frac{1}{2n-1} > \frac{1}{2n}$$

for each positive integer n. This contradiction concludes the proof. ■

Proof 9

This proof without words compares H_n and $\int_1^{n+1} 1/x\,dx$. Its variations, including those involving the integral test, are among the most popular proofs of the divergence of the harmonic series.

Proof:



$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

■

Proof 10

While not completely rigorous, this proof is thought-provoking nonetheless. It may provide a good exercise for students to find possible flaws in the argument.

Proof:

$$\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots &= \int_0^1 (1 + x + x^2 + \cdots + x^{n-1} + \cdots)\,dx \\
&= \int_0^1 \left(\sum_{k=0}^{\infty} x^k\right) dx \\
&= \int_0^1 \left(\frac{1}{1-x}\right) dx \\
&= \infty.
\end{aligned}$$

■

Proof 11

The proof above is very similar to one given by Euler in 1748 in his *Introductio in analysin infinitorum*. Euler first established a series representation for $\ln(1-x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots.$$

Having done so, his proof was simple, though it hardly meets today's standards of rigor.

PROOF: Start by writing $\ln(1-x)$ as a power series:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots.$$

It follows that

$$\ln 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right).$$

■

Proof 12

The following proof was given by Jacob Bernoulli in his 1689 *Tractatus de seriebus infinitis*. The presentation here is similar to that given by Dunham (1999, page 30).

PROOF: First notice that if c is an integer and $c > 1$, then

$$\frac{1}{c+1} + \frac{1}{c+2} + \cdots + \frac{1}{c^2} \geq \left(c^2 - c\right)\frac{1}{c^2} = 1 - \frac{1}{c}.$$

Now add $1/c$ to the left-hand and right-hand sides to establish that

$$\frac{1}{c} + \frac{1}{c+1} + \frac{1}{c+2} + \cdots + \frac{1}{c^2} \geq 1.$$

It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{25}\right) + \left(\frac{1}{26} + \cdots + \frac{1}{676}\right) + \cdots \\ &\geq 1 + 1 + 1 + 1 + \cdots. \end{aligned}$$

■

Proof 13

Although Jacob Bernoulli also gave this next proof, he credited its discovery to his brother Johann. An enjoyable account of the history of the proof can be found in the works of Dunham (1987, 1990). We present a modern version of the Bernoulli proof.

Proof: Consider the series

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

As it is written on the right, the series is telescoping and converges to 1. With this series serving as an illustration, note that

$$\sum_{n=k}^{\infty}\frac{1}{n(n+1)}=\sum_{n=k}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{k},\quad k=1,2,3,\ldots.$$

Now suppose that the harmonic series converges with sum S. Then

$$\begin{aligned}
S &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \\
&= 1+\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}+\frac{7}{56}+\cdots \\
&= 1+\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\right)+\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots\right) \\
&\qquad +\left(\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots\right)+\cdots \\
&= 1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=2}^{\infty}\frac{1}{n(n+1)}+\sum_{n=3}^{\infty}\frac{1}{n(n+1)}+\cdots \\
&= 1+1+\frac{1}{2}+\frac{1}{3}+\cdots \\
&= 1+S.
\end{aligned}$$

The contradiction $S=1+S$ concludes the proof. ■

Proof 14

Here is an unusual proof by contradiction. In this proof, we examine the subsequence $\{H_{T(n)}\}$, where $T(n)=n(n+1)/2$ is the nth triangular number.

PROOF: Suppose that the harmonic series converges with sum S. Then S must be greater than 2 since $H_4 = 25/12$. Now notice that

$$\begin{aligned}
S &= 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)+\left(\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right) \\
&\quad +\left(\frac{1}{11}+\cdots+\frac{1}{15}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{21}\right)+\cdots \\
&> 1+\frac{2}{3}+\frac{3}{6}+\frac{4}{10}+\frac{5}{15}+\frac{6}{21}+\cdots \\
&= \frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\cdots \\
&= 2\sum_{n=2}^{\infty}\frac{1}{n} \\
&= 2(S-1).
\end{aligned}$$

The inequality $S > 2(S-1)$ implies $S < 2$. This contradiction concludes the proof. ■

Proof 15

Several of the proofs in this article have been based on a common theme: if the sequence $\{\sigma_n\}$ grows fast enough, then the corresponding subsequence of the harmonic series $\{H_{\sigma_n}\}$ is bounded below by a linear function of n. This theme makes another appearance in the following proof.

PROOF: First notice that if k is an integer and $k > 1$, then

$$\frac{1}{(k-1)!+1}+\frac{1}{(k-1)!+2}+\cdots+\frac{1}{k!} > \frac{k!-(k-1)!}{k!} = 1-\frac{1}{k}.$$

Now consider the subsequence $\{H_{n!}\}$.

$$\begin{aligned}
H_{n!} &= \sum_{k=1}^{n!}\frac{1}{k} = 1+\sum_{k=2}^{n}\left(\frac{1}{(k-1)!+1}+\cdots+\frac{1}{k!}\right) \\
&> 1+\sum_{k=2}^{n}\left(1-\frac{1}{k}\right) \\
&= 1+\sum_{k=1}^{n}\left(1-\frac{1}{k}\right) \\
&= 1+n-H_n.
\end{aligned}$$

It follows that $2H_{n!} > H_{n!} + H_n > n + 1$. Therefore $\{H_{n!}\}$ is unbounded, and the harmonic series diverges. ■

More (or Less) Proofs

The divergence of the harmonic series is sometimes proved by appealing to more general results. The most common examples of this involve the integral test or the p-series test. In this section, we present a number of often overlooked results, each of which immediately implies the divergence of the harmonic series.

Proof 16

The following modified nth term test quickly shows that the harmonic series cannot converge.

> *Suppose* $\{a_n\}$ *is both positive and decreasing. If* $\sum a_n$ *converges, then* $\lim_{n\to\infty} n\, a_n = 0$.

Goar (1999) describes an episode in the history of analysis in which Louis Olivier used the converse of this result as a convergence test. Olivier's mistake was corrected by 26-year-old Neils Abel, shortly before Abel's untimely death.

Proof 17

After learning the p-series test, students often believe that the harmonic series forms a kind of boundary between the convergent and divergent series. Fortunately, it is easy to find counterexamples. The next result shows that a divergent series can actually be very much less (term-by-term) than the harmonic series. A simple proof of this result was given by Ash (1997). The divergence of the harmonic series follows by setting $d_n = 1$ for each n.

> *Suppose* $\sum_{n=1}^{\infty} d_n$ *is a divergent series with positive terms. If* $s_n = d_1 + d_2 + \ldots + d_n$, *then* $\sum_{n=2}^{\infty} d_n/s_{n-1}$ *diverges.*

Proof 18

Even though infinite products are rarely discussed in first-year calculus, it is natural for students to question their existence and convergence. This next result could easily be included in a short introduction to the topic. It is similar to Proof 5, and it shows that products and sums share at least one very important property.

> *Suppose $\{a_n\}$ is a sequence with nonnegative terms. Then $\sum a_n$ and $\prod(1+a_n)$ either both converge or both diverge.*

Proof 19

The next result has been proposed as an alternative to the integral test (Jungck 1983).

> *Let f be positive and nonincreasing on $[k,\infty)$, k a positive integer, and let g be any antiderivative of f. Then $\sum_{n=k}^{\infty} f(n)$ converges if and only if g is bounded above on $[k,\infty)$.*

The proof of this result provides an interesting application of the mean value theorem. Note that the integral test follows immediately by letting $g(x) = \int_k^x f(t)\,dt$.

Proof 20

While this result is the least elementary of our collection, we include it because of its relative obscurity and its possible appeal to students. The test is due to Abu-Mostafa (1984), and it has a very nice geometric interpretation. The divergence of the harmonic series follows by setting $f(x) = x$.

> *Let f be a real function such that d^2f/dx^2 exists at $x = 0$. Then $\sum_{n=1}^{\infty} f(1/n)$ converges absolutely if and only if $f(0) = f'(0) = 0$.*

Conclusion

The harmonic series has diverged again and again for well over six centuries. In this article, we surveyed a number of the divergence proofs, but there are others. We chose to highlight those that are particularly accessible to first-year calculus students. In our experience, these proofs have excited and motivated students. Some proofs have provided historical contexts, and some have provided connections to other topics. In any case, as we see it, there are at least $H_{1000000}$ good reasons to share these proofs with your students.

References

Abu-Mostafa, Y. S. (1984). A differentiation test for absolute convergence. *Mathematics Magazine 57*(4), 228–231.

Ash, J. M. (1997). Neither a worst convergent series nor a best divergent series exists. *College Mathematics Journal 28*(4), 296–297.

Cohen, T. and W. J. Knight (1979). Convergence and divergence of $\sum_{n=1}^{\infty} 1/n^p$. *Mathematics Magazine 52*(3), 178.

Cusumano, A. (1998). The harmonic series diverges. *American Mathematical Monthly 105*(7), 608.

Dunham, W. (1987). The Bernoullis and the harmonic series. *College Mathematics Journal 18*(1), 18–23.

Dunham, W. (1990). *Journey Through Genius: The Great Theorems of Mathematics*. John Wiley and Sons.

Dunham, W. (1999). *Euler: The Master of Us All*. The Mathematical Association of America.

Ecker, M. W. (1997). Divergence of the harmonic series by rearrangement. *College Mathematics Journal 28*(3), 209–210.

Fleron, J. F. (1999). Gabriel's wedding cake. *College Mathematics Journal 30*(1), 35–38.

Goar, M. (1999). Olivier and Abel on series convergence: An episode from early 19th century analysis. *Mathematics Magazine 72*(5), 347–355.

Honsberger, R. (1976). *Mathematical Gems II*. The Mathematical Association of America.

Johnson, P. B. (1955). Leaning tower of lire. *American Journal of Physics 23*(4), 240.

Jungck, G. (1983). An alternative to the integral test. *Mathematics Magazine 56*(4), 232–235.