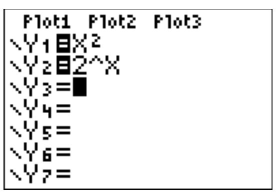
Order of Magnitude of a Function

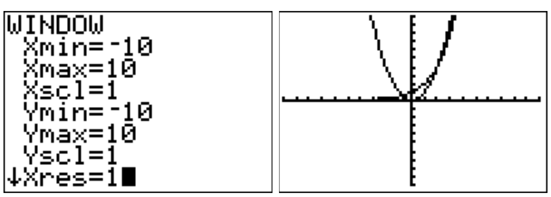
One of the competencies listed in the AP Calculus Course Description is "Comparing relative magnitudes of functions and their rates of change. (For example, contrasting exponential growth, polynomial growth, and logarithmic growth.)" Through the use of graphing technology we can explore functions and their rates of change through graphs and tables.

In 1995, Free Response Questions AB4-BC2 involved the graphs of $f(x) = x^2$ and $g(x) = 2^x$. Students were shown a graph of the two functions, not to scale, and were *told* that the graphs of the two functions intersected three times. Students were required to find the three points of intersection. Many students only found two of the three points because the third point was outside the standard window. A student who understands "order of magnitude" will recognize that exponential functions have a higher order of magnitude than power functions and will eventually be larger than any power function. Therefore, for large values of *x*, the function *g* will be larger than the function *x*.

One way to illustrate this feature is to adjust the viewing window in several different ways to highlight different aspects of the functions. In each case write down a feature that is emphasized using the given window.

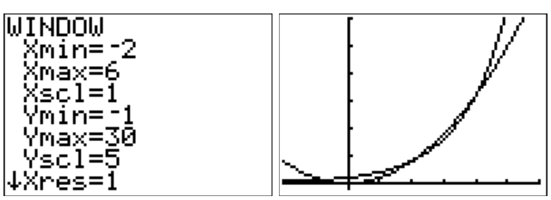


Case 1

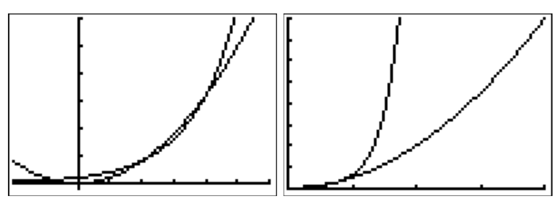


Case 2

Case 3



Case 4



One way to classify functions according to "order of magnitude" is to limits of a ratio.

Let L be the $\lim_{x\to\infty} \frac{f(x)}{g(x)}.$ If L is infinite, then $f(x)$ has a higher order of magnitude than $g(x).$ If $L = 0,$ then $f(x)$ has a lower order of magnitude than $g(x).$ If L is a finite, nonzero number, then $f(x)$ has the same order of magnitude as $g(x).$

Example 1. Rank each of the following functions in increasing order according to its order of magnitude.

i. Power function: $f(x) = x^n$

ii. Logarithmic function: $g(x) = \ln x$

iii. Exponential function: $h(x) = e^x$

Example 2. Without using L'Hopital's Rule, evaluate the following limits. Justify your work.

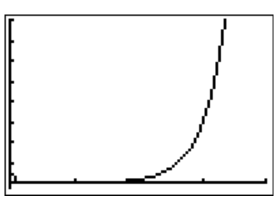
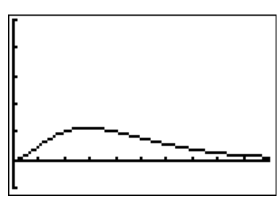
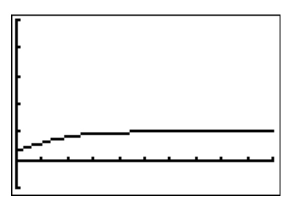
i. $\lim_{x\to\infty} \frac{\ln 3x}{x^3}$

ii. $\lim_{x\to\infty} \frac{x^{100}}{e^{0.01x}}$

iii. $\lim_{x\to\infty} \frac{\sqrt{x}}{x}$

Still another way to determine relative order of magnitude is to view the graph of the ratio of two functions $f(x)$ and $g(x).$

Example 3. The graph of $\frac{f(x)}{g(x)}$ is shown. State whether $f(x)$ or $g(x)$ has the higher order of magnitude.

i.  ii.  iii. 

Example 4. How many times do the graphs of $y = 2^x$ and $y = x^2$ intersect?

Example 5. How many times do the graphs of $y = 2^x$ and $y = x^{100}$ intersect?