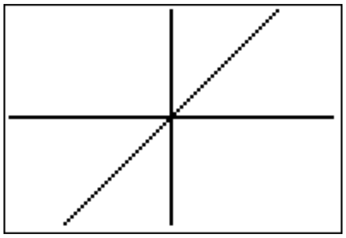
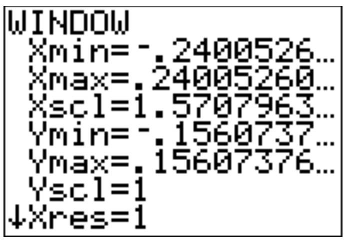
# Introduction to Taylor Series

1. Consider the function shown below. What function do you think it is? _________



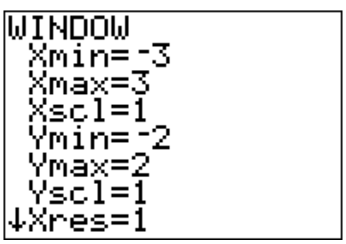
With the calculator set to ExpressionOff watch as we trace along the function. Do the values shown support your choice of function? ____ Explain_________

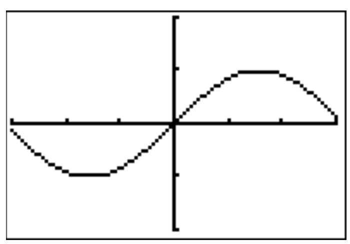
The function shown had the following window.




Is this window large enough to determine a function? ______________

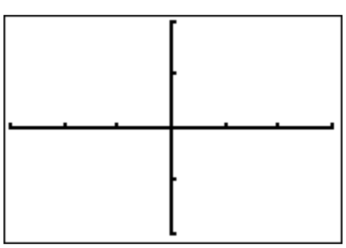
ZoomOut to the following window



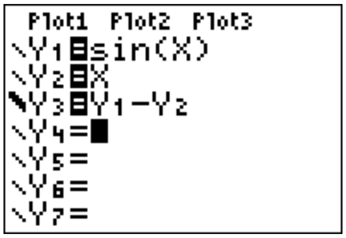
What function does the graph now seem to resemble? _________________

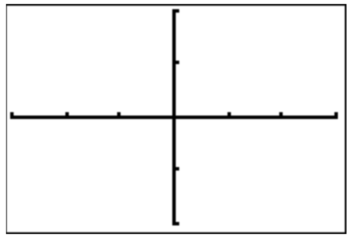
Let's plot $y = \sin x$ and $y = x$ on the same axis below.



Mark the intervals where the two functions seem to approximate each other. We can see that the difference between the two functions near the origin is close to zero. But as we move away from the origin, the differences get larger.

Let's take a closer look at the difference between the functions. Plot the following function in the previous window. Notice that $Y_3$ is in thickline mode



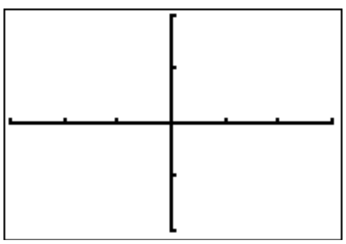
What type of function does the difference resemble? ______________

Enter your guess in $Y_4$. How well does it approximate the difference? _____ Adjust your results by using a constant of multiplication. What is your adjusted difference function?

___________. Notice that $y = -\frac{x^3}{6}$ does a good job of approximating the graph in the window.

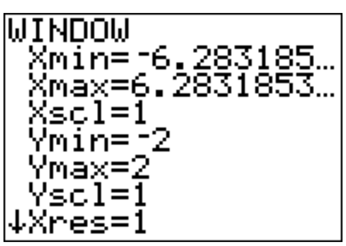
We now have $\sin x - x \approx -\frac{x^3}{6}.$ Rewrite the equation to isolate $\sin x$. ___________________

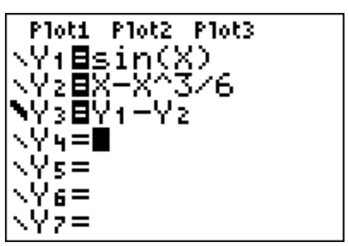
Let's look at the graphs of the two functions we now have. Sketch the results in our window.

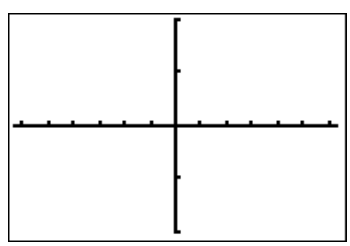


Mark the intervals where the two functions seem to approximate each other. Use the TABLE feature of the calculator to find the interval where the two functions are equal to each other within three decimal places. ______________________

II. Let's repeat the process by enlarging the window as shown below and sketching the difference between the two functions as shown



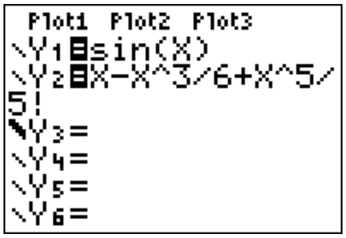
What familiar function does the difference graph resemble? (Hint: It is not a cubic function.)

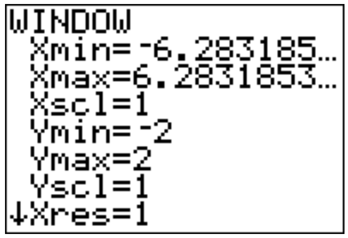
_______________________________

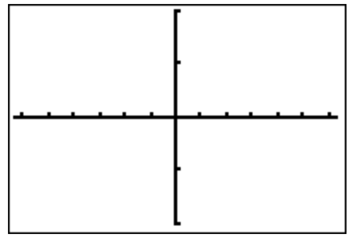
Enter your guess in Y4. How well does it approximate the difference? _____ Adjust your results by using a constant of multiplication. What is your adjusted difference function? ___________.

Notice that $y_4 = \frac{x^5}{120}$ does a good job of approximating the graph in the window. We now have $\sin x - \left(x - \frac{x^3}{6}\right) \approx \frac{x^5}{120}$. Rewrite the equation to isolate $\sin x$. ___________________

Graph the results as shown in the given window.



Mark the intervals where the two functions seem to approximate each other. Use the TABLE feature of the calculator to find the interval where the two functions are equal to each other within three decimal places. ______________________

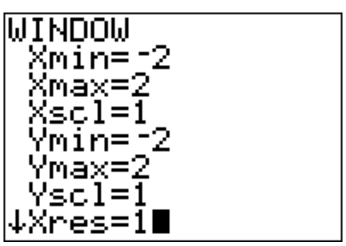
III. Use the pattern to write a 13th degree polynomial that can be used to approximate $y = \sin x$.

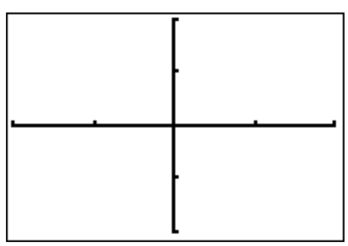
Enter this function in Y2 along with $y = \sin x$ in Y1. Use your calculator to find $\sin(0.5)$. Write the result. ___________ Now use your calculator to evaluate $y_2(0.5)$. Write the result _______________________.

You can see that the polynomial function does an excellent job in approximating the value of the transcendental function. Generally, it is easier to work with polynomial functions instead of transcendental functions. We want to see if we can develop polynomial lookalike functions for other functions in a manner similar to what we did with $y = \sin x$.

IV. Let's look at $y = \cos x$.

Sketch the graph in the window with the following values.



What familiar polynomial function does this seem to resemble? ______________

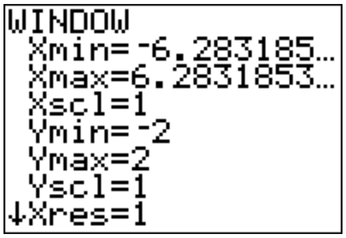
We could repeat the same process as we did for $y = \sin x$. But, it would be nicer if we could find a simpler method. Write down your final polynomial approximation to $y = \sin x$.

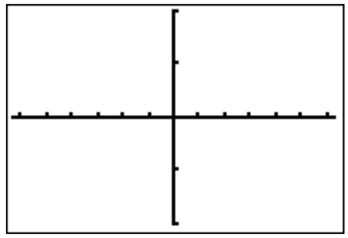
_____________________

How does $\cos x$ relate to $y = \sin x$? ______________________

Write the result $\cos x \approx$ ____________________________________

Now graph $y = \cos x$ and your polynomial approximation in the following window.



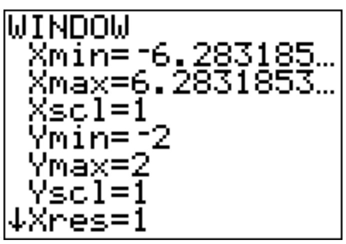



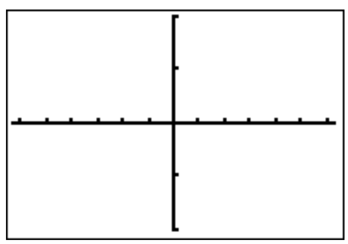
Mark the intervals where the two functions seem to approximate each other. Use the TABLE feature of the calculator to find the interval where the two functions are equal to each other within three decimal places. ___________________________

As you can see, it is possible to get new polynomials from old polynomials. This is a recurring theme throughout our study of series.

How could you find a polynomial approximation for $y = \cos\left(\frac{1}{2}x\right)$?

Sketch the graphs of $y = \cos\left(\frac{1}{2}x\right)$ and its polynomial approximation in the following window.



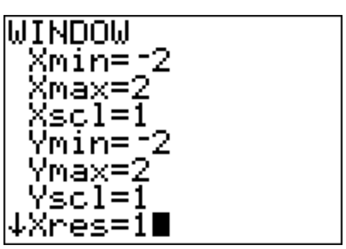



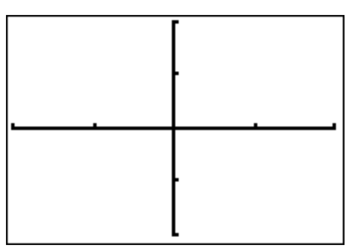
V. Let's see if we can find a polynomial representation for $y = e^x$.

To begin, we want to write the equation of a first degree polynomial with the same slope and y-intercept as $y = e^x$ at the point (0, 1).

Write the equation of the line tangent to the graph of $y = e^x$ at (0, 1). _________________

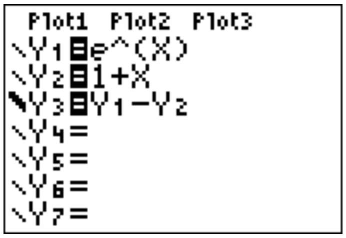
Graph the function and the tangent line in the following window.

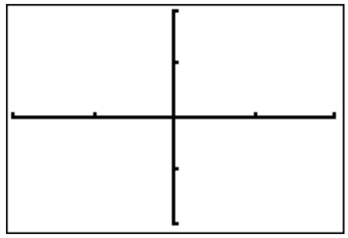


How well does the line approximate the function in the window? ______________

Let's graph the difference between the functions as we did before. Note the thick line style for $Y_3$.



What shape does the error graph seem to resemble? ____________ What are the obvious limitations to this model

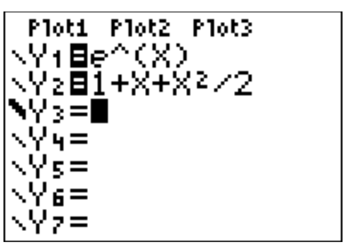
Although the difference doesn't exactly resemble a parabola, we are trying to use polynomial representations. Does $y = x^2$ appear to be the correct model? _____

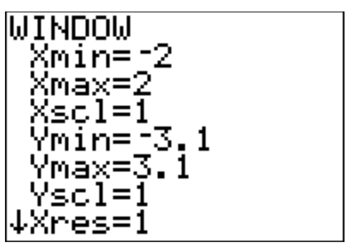
Obviously, we need to divide by some constant to widen the graph. What would you recommend? Let's try $y = \dfrac{x^2}{2!}$. Enter this function in $Y_4$. How well does it match the actual error? ___________________________

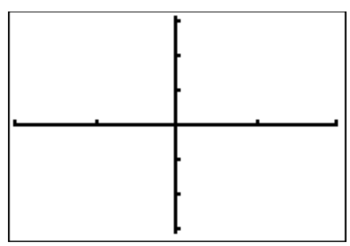
Notice that the polynomial works well in the center of the graph but doesn't seem to work well near the edges of the window. We ended up with $e^x - (1 + x) \approx \dfrac{x^2}{2!}$. Solve this equation for $e^x$.

___________________________

Sketch the function and its polynomial approximation in the window.



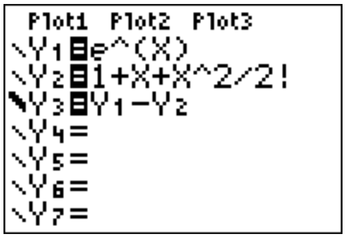



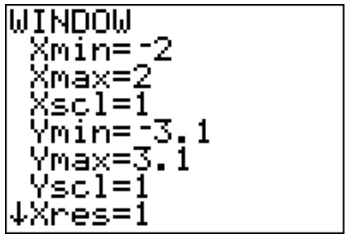



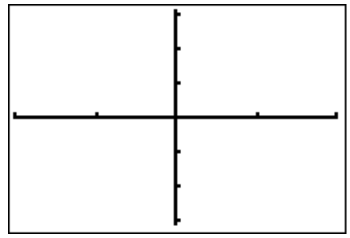
Mark the intervals where the two functions seem to approximate each other. Use the TABLE feature of the calculator to find the interval where the two functions are equal to each other within three decimal places. ______________________

Once again, notice that the polynomial works well in the center of the graph but doesn't seem to work well near the edges of the window.

Let's look at the differences again. Note the thick line style for $Y_3$.



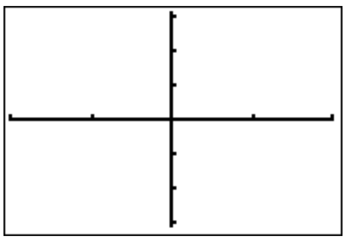



This difference looks like a cubic polynomial. Based on our previous work, what coefficient do you think the cubic function will have?_____________

This leads to the relationship $e^x - (1 + x + \frac{x^2}{2!}) \approx \frac{x^3}{3!}$.

Solve this equation for $e^x$. ___________________________

Graph the two functions and see how well they match.



Mark the intervals where the two functions seem to approximate each other.

What do you notice about the interval of matching as the number of terms increases?

___________________________

Write the first 8 terms for a polynomial approximation for $y = e^x$.

Use your calculator to find the value of $e^2$. _____________

Use the polynomial to approximate $e^2$. _____________

Notice that the polynomial approximation doesn't approach $e^2$ as rapidly as the sine or cosine functions did.

VI. Power Series

From your pre-calculus courses you learned that you can represent a repeating decimal as the sum of an infinitely decreasing geometric series. For example,

$\overline{0.9} = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$ or $\overline{0.9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$ which can be written as $\frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$. This expression can be rewritten as $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^{n-1}$.

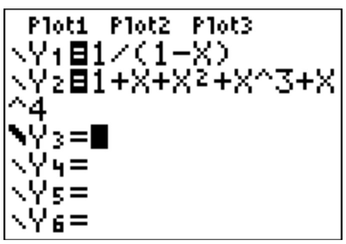
In general $\sum_{n=1}^{\infty} ar^{n-1} = a + ar^1 + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$. Recall that the sum of an infinitely decreasing geometric series can be computed by the formula $S = \frac{a_0}{1-r}$ which is valid only when $|r| < 1$.

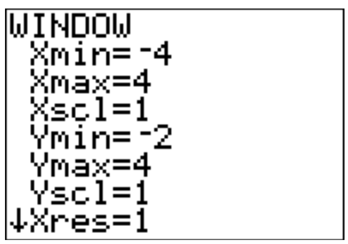
A. Creating a New Series

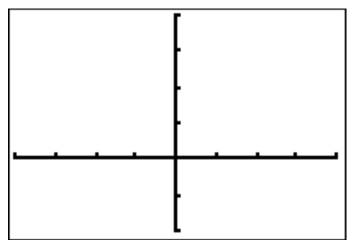
Let's generalize this formula by substituting 1 for a and x for r. Write the first five terms of the expansion.

$\frac{1}{1-x} =$ ________________________________

While this may seem difficult to believe, a look at the graphs can support our prediction. Sketch the graph of the function and its approximation in the window below.



This is a fascinating discovery. Not only can we use polynomials to approximate transcendental functions, we can approximate rational functions as well.

Mark off the interval where the two functions seem to be equal.

Now add a few more terms to $Y_2$ to see how the interval of convergence changes. Is there much change? _____________

Add some more terms. Notice that the window of convergence doesn't seem to get much wider. Will the right hand branch of the rational function ever get "covered" by the polynomial approximation? Explain_______________________________________

When we used polynomials to approximate trig functions, we noticed the window of convergence widened each time we added additional term to the polynomial. As you can see, that is not always the case. What will be the interval of convergence for the function $f(x) = \dfrac{1}{1-x}$? ______________________________________________

B. Looking at other functions

We can use the basic template for the rational function $\dfrac{1}{1-x}$ to write other rational functions in series mode.

Consider $y = \dfrac{1}{1+x}$. Rewrite this function using $\dfrac{1}{1-x}$. ________________

How will this change the series expansion? ________________________

Write the first five terms for the series expansion for $y = \dfrac{1}{1+x}$. ____________________

Use the ideas of these two rational functions to write a series expansion for $y = \dfrac{1}{1+x^2}$.

___________________________________________________________________________

Where have you seen $\dfrac{1}{1-x}$ before? __________________

How could you use this idea to write a new power series for a familiar function?