Geogebra: Logistic Model

Name: Block: Seat:

We want to make a GeoGebra worksheet to explore the solution to

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

where $y(t)$ is the population as a function of time t, k is the coefficient of growth, and L is the **carrying capacity**.

Theorem: The solution to the differential equation is

$$y(t) = \frac{L}{1 + be^{-kt}}$$

where b is some constant of integration.

1. Make 3 sliders: L, b, and k with the **Slider Tool** (2nd from right). You can edit these later by selecting "Object Properties either in the graph view or the Algebra view.
   (a) Name: "L", Min:1, Max: 60, Increment: 1
   (b) Name: "b", Min:0, Max: 40, Increment: 0.5
   (c) Name: "k", Min:0, Max: 0.2, Increment: 0.01
2. In the "Input" field type: "`M: y = L`" to make the asymptote. You can edit it be a different color and dashed if you would like.
3. In the "Input" field type: "`P: y = L/(1+b*e^ (-1*k*x) )`" to make the population function. You can edit it be a different color if you would like.
4. In the "Input field type "`I: (0, P(0) )`" to make a point to show the population at time 0 (the initial population). Edit it so that under **Show Label** you select "Name and Value" or if you prefer, just "Value"
5. You can use the **Move Graphics View** button on the right to change the scale, but remember to switch back to **Move** tool when you are done to change the values on the sliders.
6. Try to find a value for b so that y(0)=6, when the $L = 60$.
7. Try to make a new line for $y = L/2$ and a new Point defined as the intersection of this new line and population function.
8. You can save your work if you make a free account at `http://geogebra.org`.
9. A population of rabbits in a large field is given by the formula

$$P(t) = \frac{1000}{1 + 121.51e^{-0.7t}}$$

   where t is the number of months after some rabbits were released into the field.
   (a) Determine the maximum number of rabbits this field can maintain.
   (b) What is the value of k?
   (c) What is the population when $t = 0$?
   (d) What is the population when the rate of growth is greatest?
   (e) Use your calculator to determine when the population reaches the value found in part (d).

While the Larson text book uses

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

if you watch the **First order differential equations** videos at `https://www.khanacademy.org/math/differential-equations` then you will see Sal prefers the notation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where K is the carrying capacity, N is the number (or population size) and r is the coefficient of growth.

One could also describe the logistic equation as

$$\frac{dP}{dt} = kP(M - P)$$

where the M is the maximum carrying capacity and P is the population size, and k is some constant. Notice that the last factor is close to zero when the population P is close to the capacity M. Notice also the last factor is extreme when the population is far from the capacity.

$$\frac{dP}{dt} = kP(M - P) = kMP\left(\frac{M}{M} - \frac{P}{M}\right) \to k_2P\left(1 - \frac{P}{M}\right)$$

1. Make a new window and type: "`Slopefield (0.008y(30-y), 20)`" in the input field of the Algebra view. A nice graphics view is $x \in (0, 40)$ and $y \in (0, 60)$.

2. Turn on the **CAS** view, and in the CAS view type: "`SolveODE[ 0.008y(30-y), (0,6)]`" then a return]

3. Click the dot to the left of the solution in the CAS view to see the graph of the solution to Example 1 of the Epidemic worksheet.

4. The rate at which the number of moose in a new section of a game preserve is changing is given by

$$\frac{dP}{dt} = 0.005P(40 - P)$$

where t is given in years.

(a) If 4 moose are introduced to this new habitat, what will be the maximum number of moose, the habitat will support?

(b) When will the rate at which the moose population is growing be the greatest?

(c) A well-meaning, but misinformed person, suggests that by introducing 50 moose initially, there will be more moose available to hunt. Explain why this is faulty reasoning.

We can solve the differential equation of the Moose problem:

$$\frac{dP}{dt} = 0.005P(40 - P)$$

Separate the variables:

$$\frac{dP}{P(40 - P)} = 0.005dt$$

Or

$$\left(\frac{1}{P(40 - P)}\right) dP = 0.005dt$$

Rewrite the first factor of the left side of the equation using partial fractions:

$$\frac{1}{P(40 - P)} = \frac{A}{P} + \frac{B}{40 - P}$$

Multiply both sides by $P(40 - P)$ we get:

$$1 = A(40 - P) + B(P)$$

If $P = 0$, then $A = \frac{1}{40}$ and if $P = 40$ then $B = \frac{1}{40}$ so we have:

$$\frac{1}{P(40 - P)} = \frac{1}{40P} + \frac{1}{40(40 - P)}$$

By substituting this result we now we have:

$$\left(\frac{1}{40P} + \frac{1}{40(40 - P)}\right) dP = 0.005dt$$

Integrate each side:

$$\int \left(\frac{1}{40P} + \frac{1}{40(40 - P)}\right) dP = \int 0.005dt$$

$$\frac{1}{40} \int \left(\frac{1}{P} + \frac{1}{(40 - P)}\right) dP = 0.005 \int dt$$

$$\int \frac{dP}{P} + \int \frac{dP}{40 - P} = 0.2 \int dt$$

Find the antiderivative for each side:

$$\ln |P| - \ln |40 - P| + C_1 = 0.2t + C_2$$

Multiply by -1 and simplify:

$$\ln \left| \frac{40 - P}{P} \right| = -.2t - C_3$$

Exponentiate each side:

$$\frac{40 - P}{P} = e^{-0.2t - C_3} = e^{-0.2t} e^{-C_3} = Ce^{-0.2t}$$

$$\frac{40}{P} = 1 + Ce^{-0.2t}$$

Solve for P to get our population as a function of time t:

$$P(t) = \frac{40}{1 + Ce^{-0.2t}}$$

Since we have $P(0) = 4$ we know that $4 = \dfrac{40}{1+C}$ so that $C = 9$

$$P(t) = \frac{40}{1 + 9e^{-0.2t}}$$

5. If M is the maximum capacity, and the I is initial population, what will be the coefficient of the e term of the solution of the differential equation?

(now try Larson page 430, exercises 67-70, 71-77 odd)