Error Bounds in Power Series

When using a power series to approximate a function (usually a Taylor Series), we often want to know how large any potential error is on a specified interval.

There are two tools available to determine the error bound; the Alternating Series Error Bound and the Lagrange Error Bound.

<u>Alternating Series Error Bound</u>

When a series is alternating, the error is maximized in the next unused term evaluated at the difference between the center of the interval of convergence and the *x*-coordinate being evaluated.

Example 1: 1995 BC5 (calculator allowed)

Let *f* be the function defined by $f(x) = e^{-2x^2}$.

a) Write the first four nonzero terms and the general term for the Taylor series expansion of *f(x)* about *x = 0*.

b) Using Th 9.15, the Alternating Series Remainder Theorem, find the interval of convergence of the power series for *f(x)* about *x = 0*. Show the analysis that leads to your conclusion.

c) Let *g* be the function given by the sum of the first four nonzero terms of the power series for *f(x)* about *x = 0*. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

Example 2: 1982 BC5 (no calculator)

a) Write the first four terms and the general term for the Taylor series expansion about *x* = *0* for $f(x) = \ln(1+x)$.

b) For what values of *x* does the series in part a) converge?

c) Estimate the error in evaluating $\ln\left(\frac{3}{2}\right)$ by using on the first five nonzero terms.

d) Use the result in part a) to determine the logarithmic function whose Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} x^{2n}$.

Example 3. 2010 BC6

$$f(x)=\begin{cases}\dfrac{\cos x-1}{x^2} & for\ x\neq 0\\ -\dfrac{1}{2} & for\ x=0\end{cases}$$

The function *f*, defined above, has derivatives of all orders. Let *g* be the function defined by $g(x)=1+\int_0^x f(t)\,dt.$

a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about *x* = *0*. Use this series to write the first three nonzero terms and the general term of the Taylor series for *f* about *x* =*0*.

b) Use the Taylor series for *f* about *x* = *0* found in part a) to determine whether *f* has a relative maximum, a relative minimum, or neither at *x* = *0*. Give a reason for your answer.

c) Write the fifth degree Taylor polynomial for g about $x = 0$.

d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third degree Taylor polynomial for g about $x = 0$ to estimate $g(1)$. Explain why this estimate differ from the actual value by less than $\frac{1}{6!}$.

Lagrange Error Bound

When the Taylor series doesn't alternate, we still find the error by using the Lagrange error bound. The error is still tied to the next unused term according to $Error < \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \right|$ where $f^{(n+1)}(z)$ is the maximum value that the corresponding derivative takes on the given interval and *x* is the value of the polynomial function centered at *a*.

Example 4. 1999 BC4 (calculator allowed)

The function *f* has derivatives of all orders for all real numbers *x*. Assume that $f(2) = -3,\ f'(2) = 5,\ f''(2) = 3,$ and $f'''(2) = -8.$

a) Write the third degree Taylor polynomial for *f* about *x = 2* and use it to approximate $f(1.5).$

b) The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all *x* in the closed interval *[1.5, 2]*. Use the Lagrange error bound on the approximation to *f(1.5)* to explain why $f(1.5) \ne -5.$

c) Write the fourth degree Taylor polynomial, *P(x)*, for $g(x) = f\left(x^2 + 2\right)$ about *x = 0*. Use *P* to explain why *g* must have a relative minimum at *x = 0*.

Example 5.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let *h* be a function having derivatives of all orders for $x > 0.$ Selected values for *h* hand its first four derivatives are indicated in the table above. The function *h* and these first four derivatives are increasing on the interval $1 \leq x \leq 3.$

a) Write the first degree Taylor polynomial for *h* about *x* = *2* and use it to approximate $h(1.9).$ Is this approximation greater or less than $h(1.9)?$ Explain your answer.

b) Write the third degree Taylor polynomial for *h* about *x* = *2* and use it to approximate $h(1.9).$

c) Use the Lagrange error bound to show that the third degree Taylor polynomial for *h* about *x* = *2* approximates $h(1.9)$ with an error less than $3 \times 10^{-4}.$