DeMoivre's Theorem

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Historical Notes

- Lambert Discovered the idea at the same time, and they even corresponded about it
- Born 1667 in France, but moved to England as a boy and became a teacher
- Knew Newton & Halley
- Slept till he died in 1754

Look Ma, it's an exponent!

$$e^{i\pi} = \cos\pi + i\sin\pi$$

Log Rules, Man!

Multiply: Add the exponents (no need for "foil")

$$e^{\pi i}e^{3\pi i+2} = e^{4\pi i+2} = e^{4\pi i}e^{2} = e^{0i}e^{2} = e^{2}$$

Divide: Subtract the exponents

$$\frac{e^{2\pi/3}}{e^{\pi/4}} = e^{\frac{2\pi}{3}-\frac{\pi}{4}} = e^{\frac{8\pi}{12}-\frac{3\pi}{12}} = e^{5\pi/12}$$

Captain, We Need More Power!

$$(-1+i)^6 = \left(\sqrt{2}e^{3\pi i/4}\right)^6$$

$$= \left(2^{1/2}\right)^6\left(e^{3\pi i/4}\right)^6$$

$$= 2^3 e^{18\pi i/4}$$

$$= 8e^{9\pi i/2} = 8e^{\pi i/2}$$

$$= 8e^{\pi i/2} = 8i$$

Better than this…

$$(-1+i)^6 = (-1+i)(-1+i)(-1+i)(-1+i)(-1+i)(-1+i)$$

$$= \left(1-2i+i^2\right)\left(1-2i+i^2\right)\left(1-2i+i^2\right)$$

$$= 1-6i+15i^2-20i^3+15i^4-6i^5+i^6$$

$$= 1-6i+15(-1)-20(-i)+15(1)-6(i)+(-1)$$

$$= 1-6i-15+20i+15-6i-1$$

$$= 8i$$

Roots are powers too!

$$\sqrt{1+i} = (1+i)^{1/2}$$

$$= (2^{1/2}e^{\pi i/4})^{1/2}$$

$$= 2^{1/4}e^{\pi i/8}$$

or $2^{1/4}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

about $1.1 + 0.46i$

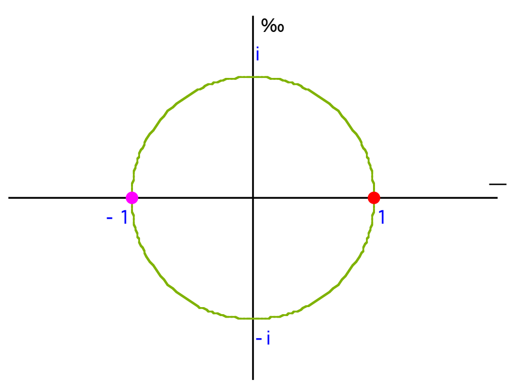
Wait! Shouldn't there be 2?

2 square roots
3 third roots
4 fourth roots, etc

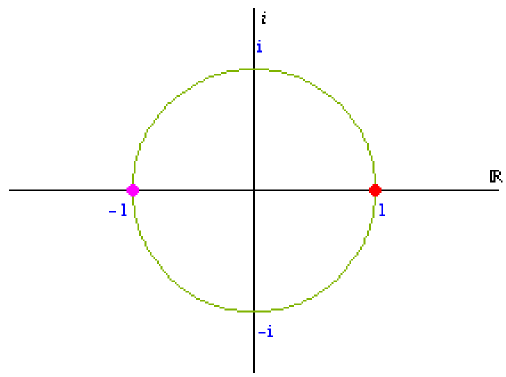
Recall the Fundamental Theorem of Algebra:
$x^n - 1 = 0$ has n roots!

$$\sqrt{4} = 2 \;\; or \;\; -2, \;\; right?$$

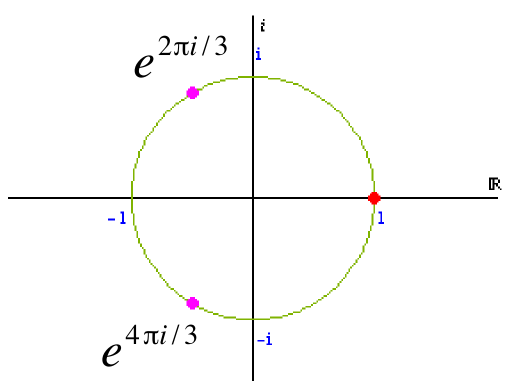
The polar view



The Square Roots of 1



The Third Roots of 1

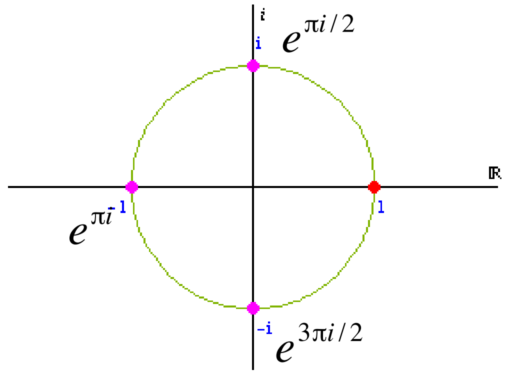


Go ahead and
Try it in your
Calculator:
(e^(2πi/3))^3
=1!
It's a third
Root of one!

One third the circle=2π/3

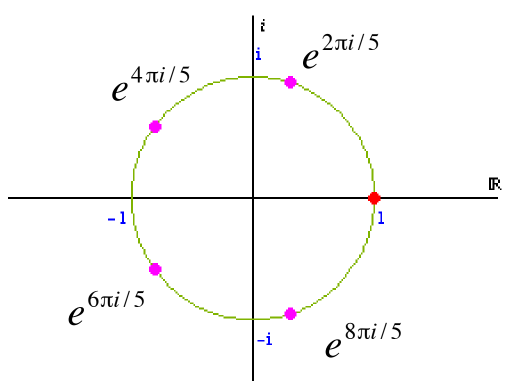
$$1, e^{2\pi i/3}, e^{4\pi i/3}$$

The Fourth Roots of 1



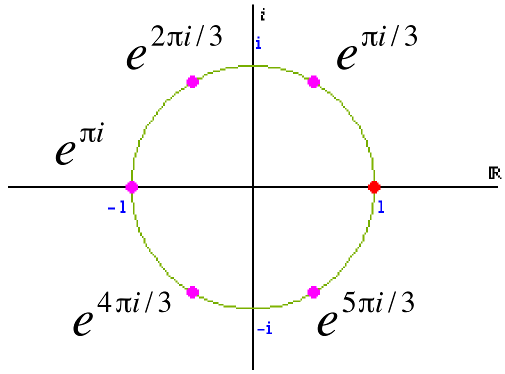
One fourth of the circle=2π/4=π/2

The fifth roots of 1



One fifth the circle=2π/5

Sixth roots of 1



One sixth of a circle=2π/6=π/3

Okay, enough with 1 already

10.3:#43. The complex cube roots of 1+*i*

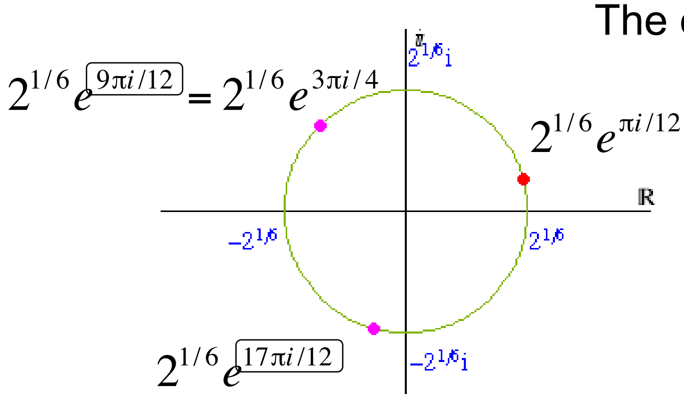
$$(1+i)^{1/3} = \left(2^{1/2} e^{\pi i/4}\right)^{1/3} = 2^{1/6} e^{\pi i/12}$$

The others? 1/3 circle away!

add $\frac{2\pi}{3} = \frac{8\pi}{12}$ *to* $\frac{\pi}{12}$!

$$\frac{\pi i}{12} + \frac{8\pi i}{12} = \boxed{\frac{9\pi i}{12}}$$

$$\frac{9\pi i}{12} + \frac{8\pi i}{12} = \boxed{\frac{17\pi i}{12}}$$



But the book says…

Yes, yes.. It works with degrees:

It will be more than 6 degrees of separation, though

$$(1+i)^{1/3} = \left(2^{1/2}(\cos 45° + i\sin 45°)\right)^{1/3} = 2^{1/6}(\cos 15° + i\sin 15°)$$

The other two? Add 120° (1/3 a circle):

$$2^{1/6}(\cos 135° + i\sin 135°)$$

$$2^{1/6}(\cos 255° + i\sin 255°)$$