Definitions of Convergent and Divergent

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot$$

If $\{S_n\}$ diverges, then the series **diverges.**

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n =$$

$$\lim_{n\to\infty} S_n =$$

The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

$$S_n =$$

$$\lim_{n\to\infty} S_n =$$

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Using partial fractions, you can write

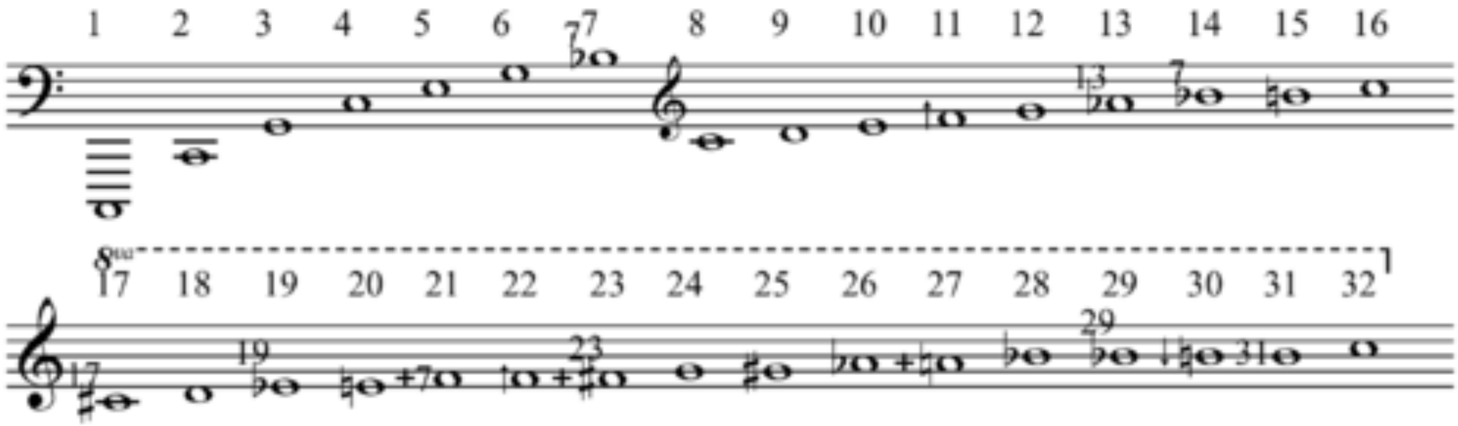
$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}.$$

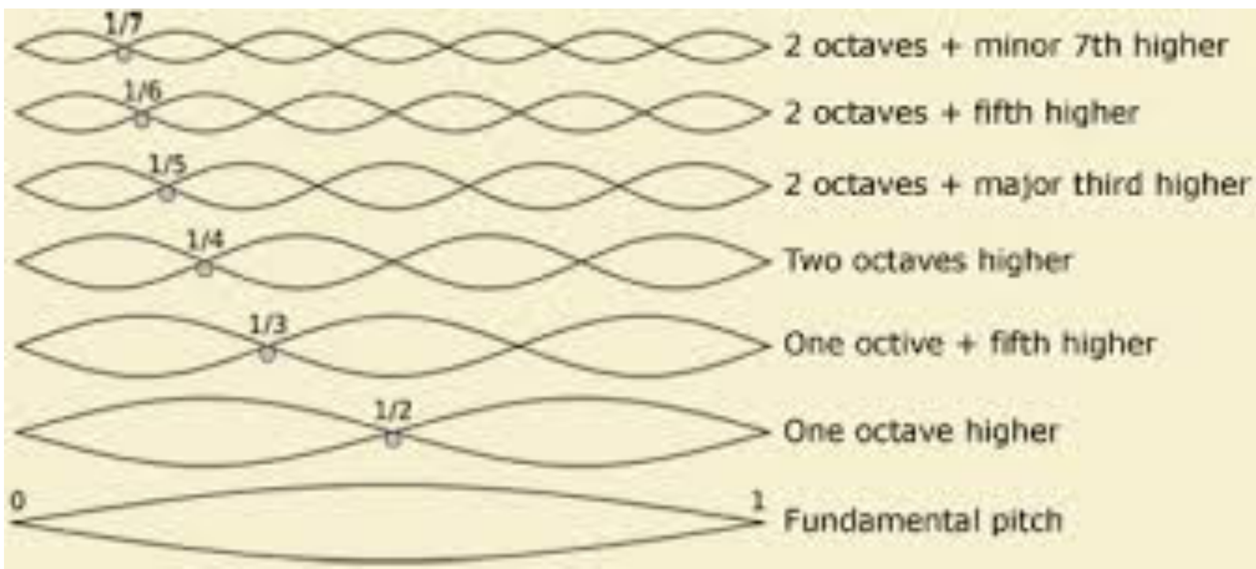
$$S_n =$$

$$\lim_{n \to \infty} S_n =$$

The Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$





https://www.youtube.com/watch?v=P81mL26Bg5k

Nicole Oresme's proof dates back to about 1350. While the proof seems to have disappeared until after the Middle Ages, it has certainly made up for lost time.

PROOF: Consider the subsequence $\{H_{2^k}\}_{k=0}^{\infty}$.

$$\begin{aligned}
H_1 &= 1 = 1 + 0\left(\frac{1}{2}\right), \\
H_2 &= 1 + \frac{1}{2} = 1 + 1\left(\frac{1}{2}\right), \\
H_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right), \\
H_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right).
\end{aligned}$$

In general,

$$H_{2^k} \geq 1 + k\left(\frac{1}{2}\right).$$

Since the subsequence $\{H_{2^k}\}$ is unbounded, the sequence $\{H_n\}$ diverges. ■

It is a classical fact that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

diverges. The standard proof involves grouping larger and larger numbers of consecutive terms, and showing that each grouping exceeds $1/2$. This proof is elegant, but has always struck me as slightly beyond the reach of students – how would one come up with the idea of grouping more and more terms together?

It turns out that one can remove this step without losing the essence of the proof:

Theorem 1. *The harmonic series diverges.*

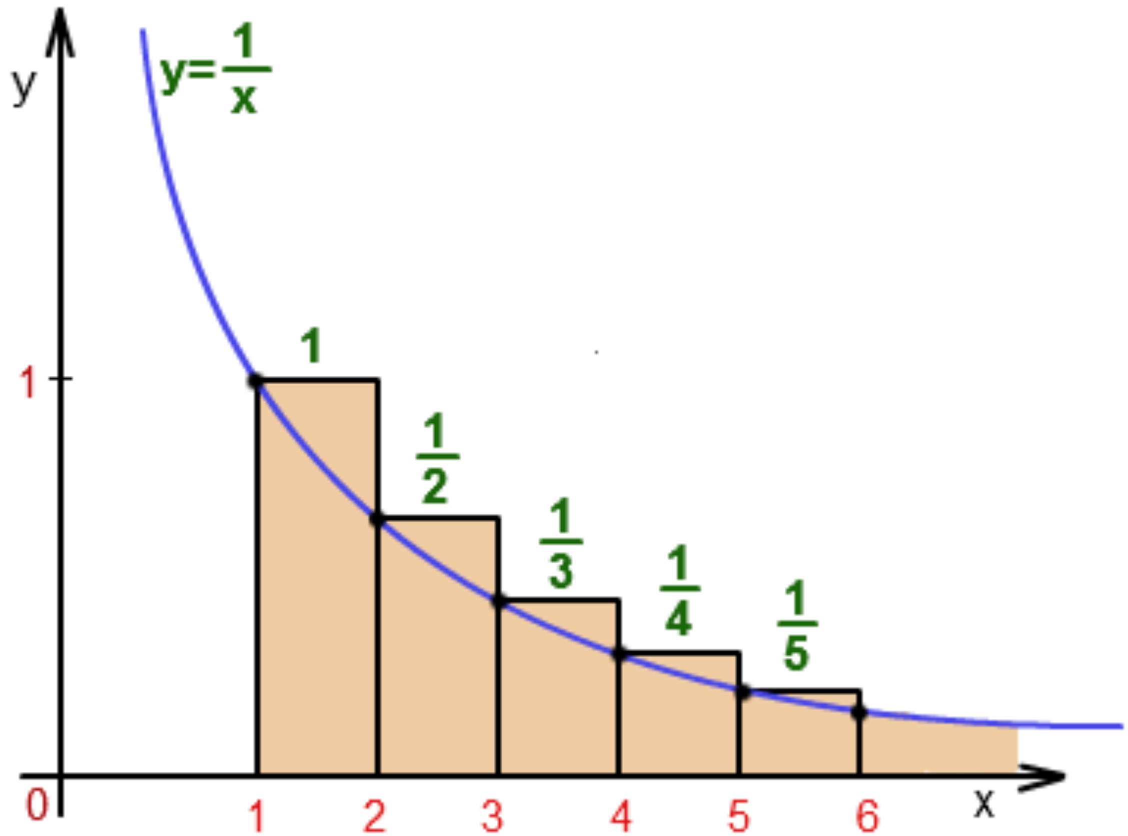
Proof. Suppose the series converges to H, i.e.

$$H=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

Then

$$\begin{aligned} H &\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}+\underbrace{\frac{1}{6}+\frac{1}{6}}+\underbrace{\frac{1}{8}+\frac{1}{8}}+\cdots \\ &= 1+\frac{1}{2}+\quad\frac{1}{2}\quad+\quad\frac{1}{3}\quad+\quad\frac{1}{4}\quad+\cdots \\ &= \frac{1}{2}+H. \end{aligned}$$

This contradiction concludes the proof. □



$$\int_1^\infty \frac{1}{x}\,dx = \left[\lim_{b\to\infty} \ln b - \ln 1\right]$$

Geometric Series

The series given in Example 1(a) is a **geometric series.** In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0$$

Geometric series

is a **geometric series** with ratio r.

For what values of r will this converge?

If $r \neq \pm 1$,

$$S_n = a + ar + ar^2 + ar^3 + \ldots$$
$$rS_n = \quad ar + ar^2 + ar^3 + \ldots$$

Theorem 9.6 Convergence of a Geometric

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Theorem 9.7 Properties of Infinite Series

THEOREM 9.7 Properties of Infinite Series

If $\Sigma\, a_n = A$, $\Sigma\, b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

Theorem 9.8 Limit of *n*th Term of a

THEOREM 9.8 Limit of *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L = \lim_{n\to\infty} S_n &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0.

Theorem 9.9 *n*th-Term Test for

THEOREM 9.9 *n*th-Term Test for Divergence

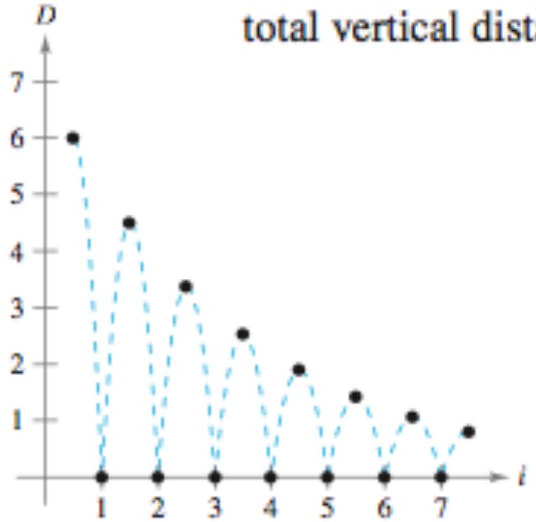
If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n} =$

b. $\sum_{n=0}^{\infty} 2^n =$

c. $\sum_{n=1}^{\infty} \frac{n!}{2n!+1} =$

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$

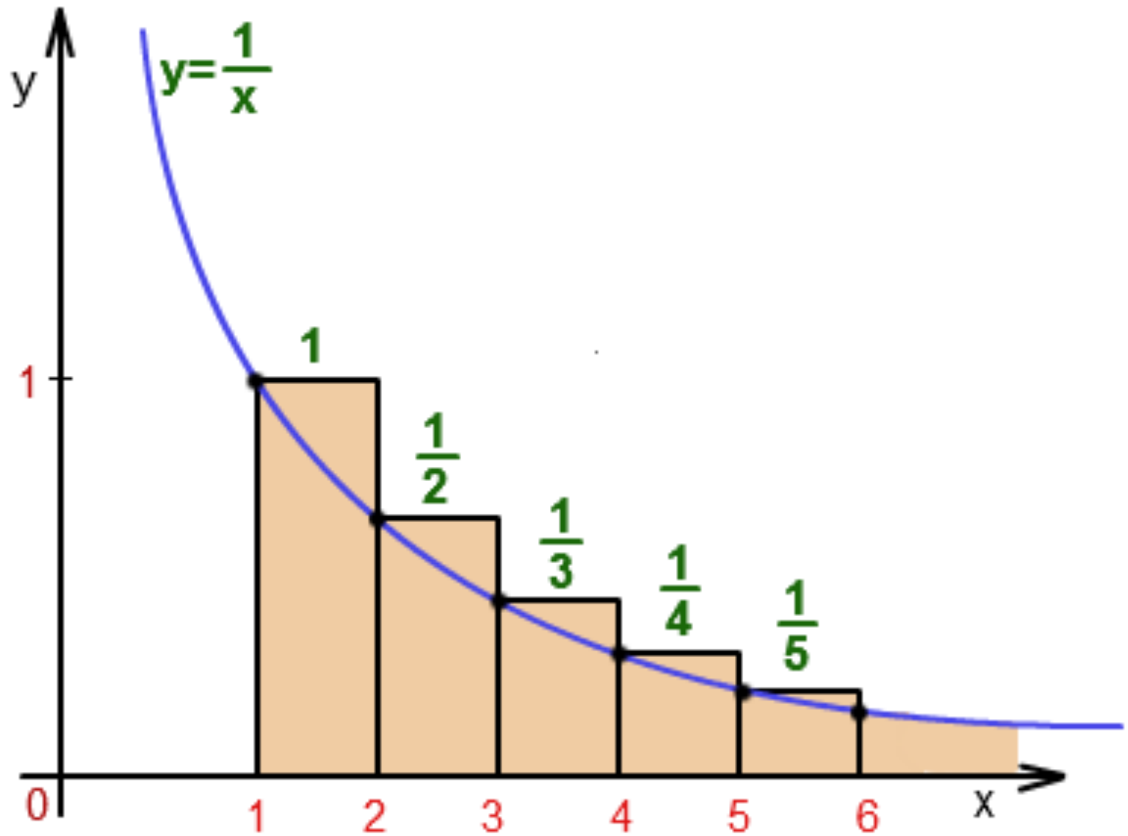
Theorem 9.10 The Integral Test

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.



$$\int_1^\infty \frac{1}{x}\,dx = \left[\lim_{b\to\infty} \ln b - \ln 1\right]$$

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

hint: u-sub

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

hint: arctan

Theorem 9.11 Convergence of *p*-Series

THEOREM 9.11 Convergence of *p*-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and

2. diverges if $0 < p \leq 1$.

p-series Proof

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Integral test for when p≠1 (Harmonic if p=1)

Conditions:
$\frac{1}{x^p}$ is non-negative when $p > 0$ and $p \neq 0$ and decreasing on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b$$

which converges if $1 - p < 0$, that is, when $1 < p$

and diverges if $1 - p > 0$, that is, when $p < 1$

Discuss the convergence or divergence of (a) the harmonic series and (b) the p-series with $p = 2$.

Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution This series is similar to the divergent harmonic series. If its terms were larger than those of the harmonic series, you would expect it to diverge. However, because its terms are smaller, you are not sure what to expect. The function $f(x) = 1/(x \ln x)$ is positive and continuous for $x \geq 2$. To determine whether f is decreasing, first rewrite f as $f(x) = (x \ln x)^{-1}$ and then find its derivative.

$$f'(x) = (-1)(x \ln x)^{-2}(1 + \ln x) = -\frac{1 + \ln x}{x^2(\ln x)^2}$$

So, $f'(x) < 0$ for $x > 2$ and it follows that f satisfies the conditions for the Integral Test. hint: let u= ln x, du = 1/x