

8 Infinite Series



8.10

Taylor and Maclaurin Series

Objectives

- Find a Taylor or Maclaurin series for a function.
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series.



Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

The following theorem gives the form that *every* convergent power series must take.

THEOREM 8.22 The Form of a Convergent Power Series

If f is represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

for all x in an open interval I containing c , then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

Taylor Series and Maclaurin Series

The coefficients of the power series in Theorem 8.22 are precisely the coefficients of the Taylor polynomials for f at c . For this reason, the series is called the **Taylor series** for f at c .

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

is called the **Taylor series** for f at c . Moreover, if $c = 0$ then the series is the **Maclaurin series** for f .

Example 1 – *Forming a Power Series*

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

and determine the interval of convergence.

Solution:

Taking successive derivatives of f yields

$$f(x) = \sin x$$

$$f(0) = \sin 0 = 0$$

$$f'(x) = \cos x$$

$$f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x$$

$$f''(0) = -\sin 0 = 0$$

$$f^{(3)}(x) = -\cos x$$

$$f^{(3)}(0) = -\cos 0 = -1$$

Example 1 – Solution

cont'd

$$f^{(4)}(x) = \sin x$$

$$f^{(4)}(0) = \sin 0 = 0$$

$$f^{(5)}(x) = \cos x$$

$$f^{(5)}(0) = \cos 0 = 1$$

and so on.

The pattern repeats after the third derivative.

So, the power series is as follows.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Example 1 – *Solution*

cont'd

$$= 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{(-1)}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

By the Ratio Test, you can conclude that this series converges for all x .

Taylor Series and Maclaurin Series

You cannot conclude that the power series converges to $\sin x$ for all x .

You can simply conclude that the power series converges to some function, but you are not sure what function it is.

This is a subtle, but important, point in dealing with Taylor or Maclaurin series.

To persuade yourself that the series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \dots$$

might converge to a function other than f , remember that the derivatives are being evaluated at a single point.

Taylor Series and Maclaurin Series

It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x -values.

If you formed the power series for the function shown in Figure 8.24, you would obtain the same series as in Example 1.

You know that the series converges for all x , and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x .

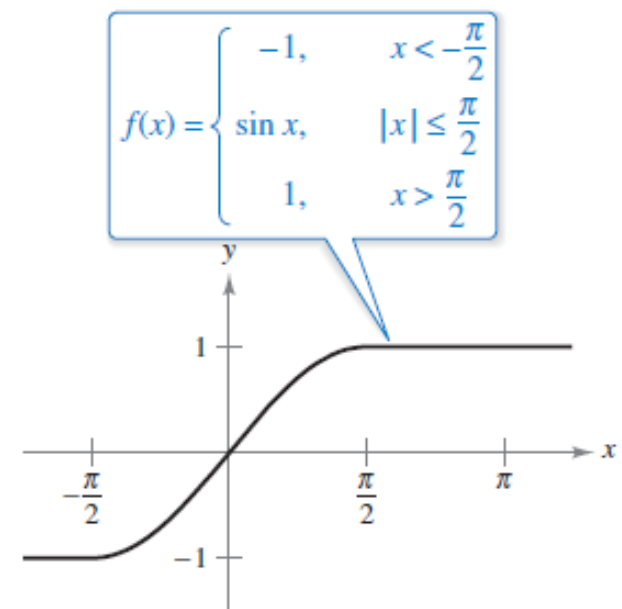


Figure 8.24

Taylor Series and Maclaurin Series

Let f have derivatives of all orders in an open interval I centered at c .

The Taylor series for f may fail to converge for some x in I . Or, even if it is convergent, it may fail to have $f(x)$ as its sum.

Nevertheless, Theorem 8.19 tells us that for each n ,

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

Taylor Series and Maclaurin Series

Note that in this remainder formula, the particular value of z that makes the remainder formula true depends on the values of x and n . If $R_n \rightarrow 0$, then the next theorem tells us that the Taylor series for f actually converges to $f(x)$ for all x in I .

THEOREM 8.23 Convergence of Taylor Series

If $\lim_{n \rightarrow \infty} R_n = 0$ for all x in the interval I , then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.$$

Example 2 – A Convergent Maclaurin Series

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x .

Solution:

You need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

is true for all x .

Example 2 – *Solution*

cont'd

Because

$$f^{(n+1)}(x) = \pm \sin x$$

or

$$f^{(n+1)}(x) = \pm \cos x$$

you know that $|f^{(n+1)}(z)| \leq 1$ for every real number z .

Therefore, for any fixed x , you can apply Taylor's Theorem (Theorem 8.19) to conclude that

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Example 2 – *Solution*

cont'd

From the discussion regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

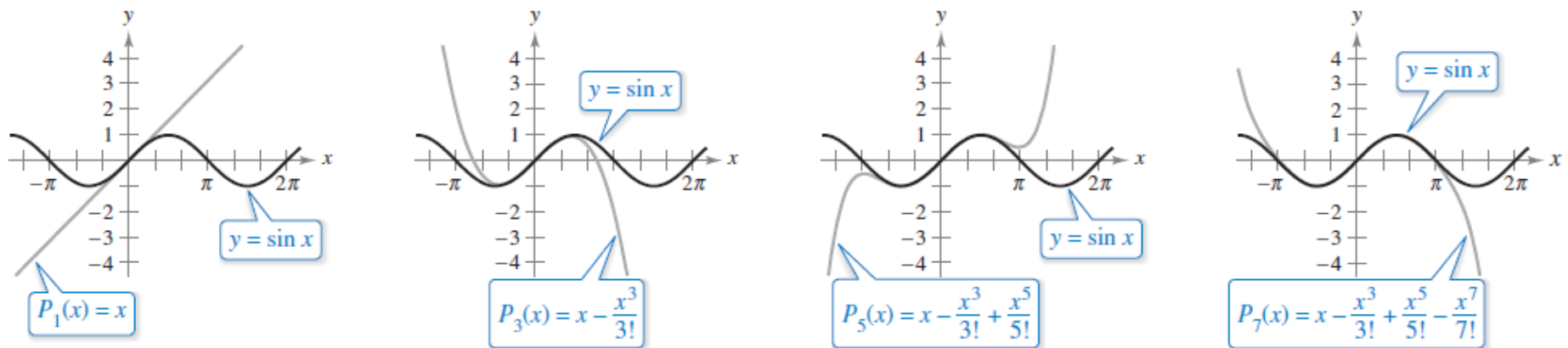
$$\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Finally, by the Squeeze Theorem, it follows that for all x , $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

So, by Theorem 8.23, the Maclaurin series for $\sin x$ converges to $\sin x$ for all x .

Taylor Series and Maclaurin Series

Figure 8.25 visually illustrates the convergence of the Maclaurin series for $\sin x$ by comparing the graphs of the Maclaurin polynomials P_1 , P_3 , P_5 , and P_7 with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



As n increases, the graph of P_n more closely resembles the sine function.

Figure 8.25

Taylor Series and Maclaurin Series

Guidelines for Finding a Taylor Series

1. Differentiate f with respect to x several times and evaluate each derivative at c .

$$f(c), f'(c), f''(c), f'''(c), \dots, f^{(n)}(c), \dots$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \dots$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.



Binomial Series

Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series**.

Example 4 – *Binomial Series*

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence.

Assume that k is not a positive integer and $k \neq 0$.

Solution:

By successive differentiation, you have

$$f(x) = (1 + x)^k$$

$$f(0) = 1$$

$$f'(x) = k(1 + x)^{k-1}$$

$$f'(0) = k$$

$$f''(x) = k(k-1)(1 + x)^{k-2}$$

$$f''(0) = k(k-1)$$

Example 4 – *Solution*

cont'd

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \quad f'''(0) = k(k-1)(k-2)$$

$\vdots$

$\vdots$

$$f^{(n)}(x) = k \cdots (k-n+1)(1+x)^{k-n} \quad f^{(n)}(0) = k(k-1) \cdots (k-n+1)$$

Which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdots + \frac{k(k-1) \cdots (k-n+1)x^n}{n!} + \cdots$$

By the Ratio Test, you can conclude that the radius of convergence is $R = 1$.

So, the series converges to some function in the interval $(-1, 1)$.



Deriving Taylor Series from a Basic List

Deriving Taylor Series from a Basic

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \dots + (-1)^n(x - 1)^n + \dots$	$0 < x < 2$
$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$	$-1 < x < 1$
$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots + \frac{(-1)^{n-1}(x - 1)^n}{n} + \dots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \dots$	$-1 \leq x \leq 1$
$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \dots$	$-1 < x < 1^*$

*The convergence at $x = \pm 1$ depends on the value of k .

Example 6 – *Deriving a Power Series from a Basic List*

Find the power series for $f(x) = \cos\sqrt{x}$.

Solution:

Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

you can replace x by $\sqrt{x}$ to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \dots$$

This series converges for all x in the domain of $\cos\sqrt{x}$ —that is, for $x \geq 0$.