8 Infinite Series

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8.7 Taylor Polynomials and Approximations

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Objectives

- Find polynomial approximations of elementary functions and compare them with the elementary functions.
- Find Taylor and Maclaurin polynomial approximations of elementary functions.
- Use the remainder of a Taylor polynomial.

Polynomial Approximations of Elementary Functions

Polynomial Approximations of Elementary Functions

To find a polynomial function P that approximates another function f, begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c).$$ Graphs of f and P pass through $(c, f(c))$.

The approximating polynomial is said to be **expanded about *c*** or **centered at *c***.

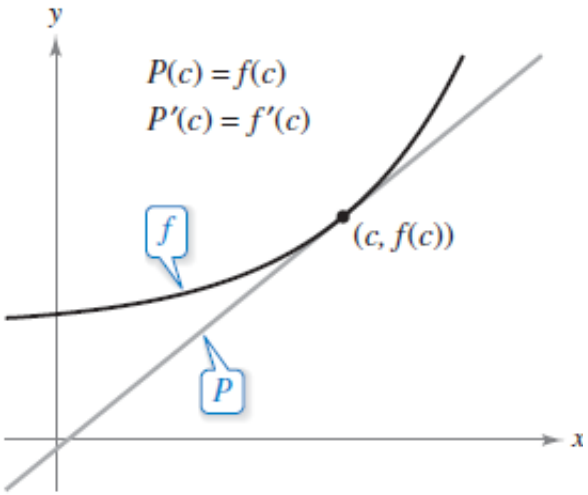
Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$.

Polynomial Approximations of Elementary Functions

Your task is to find a polynomial whose graph resembles the graph of *f* near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of *f* at the point $(c, f(c))$.

$P'(c) = f'(c)$ Graphs of f and P have the same slope at $(c, f(c))$.

With these two requirements, you can obtain a simple linear approximation of *f*, as shown in Figure 8.11.



Near $(c, f(c))$, the graph of P can be used to approximate the graph of f.

Figure 8.11

Example 1 – *First-Degree Polynomial Approximation of* $f(x) = e^x$

For the function $f(x) = e^x$, find a first-degree polynomial function

$$P_1(x) = a_0 + a_1x$$

whose value and slope agree with the value and slope of f at $x = 0$.

Solution:

Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of f, at $x = 0$, are

$$f(0) = e^0 = 1$$ Value of f at $x = 0$

and

$$f'(0) = e^0 = 1.$$ Slope of f at $x = 0$

Example 1 – *Solution*

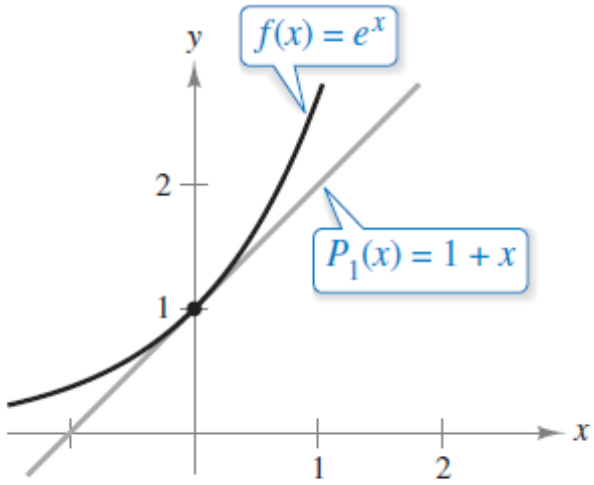
cont'd

Because $P_1(x) = a_0 + a_1x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$.

Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$.

Therefore, $P_1(x) = 1 + x$.

The figure shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$.



P_1 is the first-degree polynomial approximation of $f(x) = e^x$.

Taylor and Maclaurin Polynomials

Taylor and Maclaurin Polynomials

For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots + a_n(x - c)^n.$$

In this form, repeated differentiation produces

$$\begin{aligned}
P_n'(x) &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots + na_n(x - c)^{n-1} \\
P_n''(x) &= 2a_2 + 2(3a_3)(x - c) + \cdots + n(n - 1)a_n(x - c)^{n-2} \\
P_n'''(x) &= 2(3a_3) + \cdots + n(n - 1)(n - 2)a_n(x - c)^{n-3} \\
&\vdots \\
P_n^{(n)}(x) &= n(n - 1)(n - 2) \cdots (2)(1)a_n.
\end{aligned}$$

Taylor and Maclaurin Polynomials

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \qquad P_n'(c) = a_1, \qquad P_n''(c) = 2a_2, \quad \ldots, \qquad P_n^{(n)}(c) = n!a_n$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \qquad f'(c) = a_1, \qquad \frac{f''(c)}{2!} = a_2, \quad \ldots, \qquad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials,** named after the English mathematician Brook Taylor, and **Maclaurin polynomials,** named after the English mathematician Colin Maclaurin (1698–1746).

Taylor and Maclaurin Polynomials

Definitions of *n*th Taylor Polynomial and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **nth Taylor polynomial for f at c.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **nth Maclaurin polynomial for f.**

Example 3 – *A Maclaurin Polynomial for* $f(x) = e^x$

Find the nth Maclaurin polynomial for $f(x) = e^x$.

Solution:

The nth Maclaurin polynomial for

$$f(x) = e^x$$

is

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Remainder of a Taylor Polynomial

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy.

To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a **remainder** $R_n(x)$, defined as follows.

$$f(x) = P_n(x) + R_n(x)$$

Exact value

Approximate value

Remainder

Remainder of a Taylor Polynomial

So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|.$$

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial.

This important theorem is called **Taylor's Theorem,** and the remainder given in the theorem is called the **Lagrange form of the remainder.**

Remainder of a Taylor Polynomial

THEOREM 8.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Example 8 – *Determining the Accuracy of an Approximation*

The third Maclaurin polynomial for sin x is given by

$$P_3(x) = x - \frac{x^3}{3!}.$$

Use Taylor's Theorem to approximate sin(0.1) by $P_3(0.1)$ and determine the accuracy of the approximation.

Solution:

Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}x^4$$

where $0 < z < 0.1$.

Example 8 – *Solution*

Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.000004$$

This implies that

$$0.099833 < \sin(0.1) \approx 0.099833 + R_3(0.1) < 0.099833 + 0.000004$$

or

$$0.099833 < \sin(0.1) < 0.099837.$$