

8 Infinite Series



8.6

The Ratio and Root Tests

Objectives

- Use the Ratio Test to determine whether a series converges or diverges.
- Use the Root Test to determine whether a series converges or diverges.
- Review the tests for convergence and divergence of an infinite series.



The Ratio Test

The Ratio Test

This section begins with a test for absolute convergence—the **Ratio Test**.

THEOREM 8.17 Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. The series $\sum a_n$ converges absolutely when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. The series $\sum a_n$ diverges when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Example 1 – *Using the Ratio Test*

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Example 1 – Solution

Algebraic Solution

Because $a_n = \frac{2^n}{n!}$ you can write the following.

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1\end{aligned}$$

This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

Example 1 – Solution

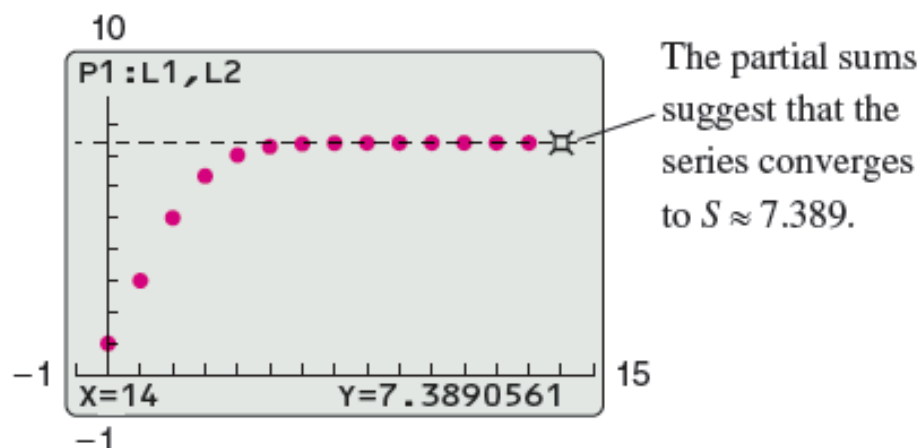
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Graphical Solution

We know that the factorial function grows faster than any exponential function.

So, you expect this series to converge.

The figure shows the first 15 partial sums of the series.





The Root Test

The Root Test

The next test for convergence or divergence of series works especially well for series involving n th powers.

THEOREM 8.18 Root Test

1. The series $\sum a_n$ converges absolutely when

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1.$$

2. The series $\sum a_n$ diverges when

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \quad \text{or} \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty.$$

3. The Root Test is inconclusive when

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1.$$

Example 4 – *Using the Root Test*

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

Example 4 – *Solution*

Algebraic Solution

You can apply the Root Test as follows.

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{e^{2n}}{n^n}} \\ &= \lim_{n \rightarrow \infty} \frac{e^{2n/n}}{n^{n/n}} \\ &= \lim_{n \rightarrow \infty} \frac{e^2}{n} = 0 < 1\end{aligned}$$

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges).

Example 4 – *Solution*

cont'd

Numerical Solution

The partial sums in the table suggest that the series converges to $S \approx 60.386$.

n	10	50	100	500
S_n	60.370	60.386	60.386	60.386



Strategies for Testing Series

Strategies for Testing Series

You have studied various tests for determining the convergence or divergence of an infinite series. Below is a set of guidelines for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the n th term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p -series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Example 5 – Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ b. $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ c. $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ e. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ f. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Example 5 – Solution

- a. For this series, the limit of the n th term is not 0 ($a_n \rightarrow \frac{1}{3}$ as $n \rightarrow \infty$). So, by the n th-Term Test, the series diverges.
- b. This series is geometric. Moreover, because the ratio of the terms $r = \pi/6$ is less than 1 in absolute value, you can conclude that the series converges.
- c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

Example 5 – Solution

cont'd

- d. The n th term of this series can be compared to the n th term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.
- e. This is an alternating series whose n th term approaches 0. Because $a_{n+1} \leq a_n$, you can use the Alternating Series Test to conclude that the series converges.
- f. The n th term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

Example 5 – *Solution*

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- g.** The n th term of this series involves a variable that is raised to the n th power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.

Strategies for Testing Series

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
n th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$ r < 1$	$ r \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum: $S = b_1 - L$
p -Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$

Strategies for Testing Series

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Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x) dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	