

8 Infinite Series



8.3

The Integral Test and p -Series

Objectives

- Use the Integral Test to determine whether an infinite series converges or diverges.
- Use properties of p -series and harmonic series.



The Integral Test

The Integral Test

THEOREM 8.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) dx$$

either both converge or both diverge.

Example 1 – *Using the Integral Test*

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

Solution:

The function $f(x) = x/(x^2 + 1)$ is positive and continuous for $x \geq 1$.

To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Example 1 – *Solution*

cont'd

So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test.

You can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2 + 1} dx &= \frac{1}{2} \int_1^{\infty} \frac{2x}{x^2 + 1} dx \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \int_1^b \frac{2x}{x^2 + 1} dx \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\ln(x^2 + 1) \right]_1^b \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} [\ln(b^2 + 1) - \ln 2] \\ &= \infty.\end{aligned}$$

So, the series *diverges*.



p -Series and Harmonic Series

p -Series and Harmonic Series

A second type of series has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

p -series

is a **p -series**, where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Harmonic series

is the **harmonic series**.

p -Series and Harmonic Series

A **general harmonic series** of the form

$$\sum_{n=1}^{\infty} \frac{1}{an + b}.$$

In music, strings of the same material, diameter, and tension, whose lengths form a harmonic series, produce harmonic tones.

THEOREM 8.11 Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges for $p > 1$, and diverges for $0 < p \leq 1$.

Example 3 – Convergent and Divergent p -Series

Discuss the convergence or divergence of (a) the harmonic series and (b) the p -series with $p = 2$.

Solution:

a. From Theorem 8.11, it follows that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \quad p = 1$$

diverges.

b. From Theorem 8.11, it follows that the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad p = 2$$

converges.