

# 8 Infinite Series



## 8.2

# Series and Convergence

# Objectives

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the  $n$ th-Term Test for Divergence of an infinite series.



# Infinite Series

# Infinite Series

One important application of infinite sequences is in representing “infinite summations.”

Informally, if  $\{a_n\}$  is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Infinite Series

is an **infinite series** (or simply a **series**).

The numbers  $a_1$ ,  $a_2$ ,  $a_3$ , and so on are the **terms** of the series. For some series, it is convenient to begin the index at  $n = 0$  (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply  $\sum a_n$ .

# Infinite Series

In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums**.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If this sequence of partial sums converges, then the series is said to converge and has the sum indicated in the next definition.

# Infinite Series

## Definitions of Convergent and Divergent Series

For the infinite series  $\sum_{n=1}^{\infty} a_n$ , the  **$n$ th partial sum** is

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums  $\{S_n\}$  converges to  $S$ , then the series  $\sum_{n=1}^{\infty} a_n$  **converges**. The limit  $S$  is called the **sum of the series**.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If  $\{S_n\}$  diverges, then the series **diverges**.

## Example 1(a) – *Convergent and Divergent Series*

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

has the partial sums listed below.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

## Example 1(a) – *Convergent and Divergent Series*<sub>cont'd</sub>

Because

$$\lim_{n \rightarrow \infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

## Example 1(b) – *Convergent and Divergent Series* cont'd

The  $n$ th partial sum of the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots$$

is

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of  $S_n$  is 1, the series converges and its sum is 1.

## Example 1(c) – *Convergent and Divergent Series* cont'd

The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$$

diverges because  $S_n = n$  and the sequence of partial sums diverges.

# Infinite Series

The series  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots$

is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots$$

Telescoping series

Note that  $b_2$  is canceled by the second term,  $b_3$  is canceled by the third term, and so on.

# Infinite Series

Because the  $n$ th partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if  $b_n$  approaches a finite number as  $n \rightarrow \infty$ .

Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n \rightarrow \infty} b_{n+1}.$$



# Geometric Series

# Geometric Series

The series  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$  is a **geometric series**.

In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots + ar^n + \dots, a \neq 0$$

Geometric series

is a **geometric series** with ratio  $r$ ,  $r \neq 0$ .

# Geometric Series

## **THEOREM 8.6** Convergence of a Geometric Series

A geometric series with ratio  $r$  diverges when  $|r| \geq 1$ . If  $|r| < 1$ , then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

### Example 3 – *Convergent and Divergent Geometric Series*

Determine the convergence or divergence of each series.

a. 
$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots$$

b. 
$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \dots$$

## Example 3 – *Solution*

### Algebraic Solution

**a.** The geometric series has a ratio of  $r = \frac{1}{2}$  with  $a = 3$ .

Because  $|r| < 1$ , the series converges and its sum is

$$S = \frac{a}{1 - r} = \frac{3}{1 - (1/2)} = 6.$$

**b.** The geometric series has a ratio of  $r = \frac{3}{2}$ .

Because  $|r| \geq 1$ , the series diverges.

# Example 3 – *Solution*

cont'd

## Numerical Solution

- a. The partial sums in the table suggest that the series converges to  $S = 6$ .

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| $n$   | 10    | 50    | 100   | 500   |
| $S_n$ | 5.997 | 6.000 | 6.000 | 6.000 |

- b. The partial sums in the table suggest that the series diverges.

|       |         |                     |                        |                        |
|-------|---------|---------------------|------------------------|------------------------|
| $n$   | 10      | 50                  | 100                    | 500                    |
| $S_n$ | 170.995 | $1.913 \times 10^9$ | $1.220 \times 10^{18}$ | $3.332 \times 10^{88}$ |

# Geometric Series

## **THEOREM 8.7** Properties of Infinite Series

Let  $\sum a_n$  and  $\sum b_n$  be convergent series, and let  $A$ ,  $B$ , and  $c$  be real numbers. If  $\sum a_n = A$  and  $\sum b_n = B$ , then the following series converge to the indicated sums.

1.  $\sum_{n=1}^{\infty} ca_n = cA$

2.  $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3.  $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$



## $n$ th-Term Test for Divergence

# *n*th-Term Test for Divergence

**THEOREM 8.8** Limit of the *n*th Term of a Convergent Series

If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

The contrapositive of Theorem 8.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the *n*th term of a series does *not* converge to 0, the series must diverge.

**THEOREM 8.9** *n*th-Term Test for Divergence

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

## Example 5 – Using the $n$ th-Term Test for Divergence

a. For the series  $\sum_{n=0}^{\infty} 2^n$ , you have

$$\lim_{n \rightarrow \infty} 2^n = \infty.$$

So, the limit of the  $n$ th term is not 0, and the series diverges.

b. For the series  $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$ , you have

$$\lim_{n \rightarrow \infty} \frac{n!}{2n! + 1} = \lim_{n \rightarrow \infty} \frac{1}{2 + (1/n!)} = \frac{1}{2}.$$

So, the limit of the  $n$ th term is not 0, and the series diverges.

## Example 5 – Using the $n$ th-Term Test for Divergence

cont'd

c. For the series  $\sum_{n=1}^{\infty} \frac{1}{n}$ , you have

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Because the limit of the  $n$ th term is 0, the  $n$ th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence.