MACLAURIN SERIES

No peaking !!!!! See if you can do the following:

	$f(x)$	Power Series Expansion:
1	$\dfrac{1}{1-x}$	
2	$\dfrac{1}{1+x}$	
3	$\dfrac{1}{1+x^2}$	
4	$\dfrac{1}{1-x^3}$	
5	$\tan^{-1} x$	
6	$\sin x$	
7	$\cos x$	
8	$\sin\left(x^2\right)$	
9	$\cos\left(\sqrt{5x}\right)$	
10	e^x	
11	e^{-x}	
12	$\ln(1+x)$	
13	$\ln(x)$ @ $x=1$	
14	$\dfrac{1}{x}$ @ $x=1$	

ANOTHER FIVE QUESTIONS:

1	Express $\int \sin\left(x^2\right)dx$ as a Power Series:
2	Estimate $\int_0^1 \sin\left(x^2\right)dx$ with an error of less than 0.001.
3	Estimate $e^{-0.1}$ using series, correct to three decimal places.
4	Find the coefficient of x^4 in the Maclaurin series for $f(x)=e^{-x/2}$.
5	Estimate $\int_0^{0.3} x^2 e^{-x^2} dx$, correct to three decimal places.

MACLAURIN SERIES (ANSWERS)

1-2	$\frac{1}{1-x}=1+x+x^2+x^3+x^4\ldots+x^n+\ldots=\sum_{n=0}^{\infty}x^n$ AND $\frac{1}{1+x}=1-x+x^2-x^3+x^4\ldots+\left(-x\right)^n+\ldots=\sum_{n=0}^{\infty}\left(-1\right)^n x^n$
3	$\frac{1}{1+x^2}=1-x^2+x^4\ldots..+\left(-x^2\right)^n+\ldots..=\sum_{n=0}^{\infty}\left(-1\right)^n x^{2n}$
4	$\frac{1}{1-x^3}=1+x^3+x^6+x^9+x^{12}\ldots..+x^{3n}+\ldots..=\sum_{n=0}^{\infty}x^{3n}$
5	$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots..+\frac{\left(-1\right)^n x^{2n+1}}{2n+1}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n+1}}{2n+1}$
6	$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots..+\frac{\left(-1\right)^n x^{2n+1}}{\left(2n+1\right)!}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n+1}}{\left(2n+1\right)!}$
7	$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots..+\frac{\left(-1\right)^n x^{2n}}{\left(2n\right)!}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{\left(2n\right)!}$
8	$\sin\left(x^2\right) = x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\ldots..+\frac{\left(-1\right)^n x^{4n+2}}{\left(2n+1\right)!}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{4n+2}}{\left(2n+1\right)!}$
9	$\cos\left(\sqrt{5x}\right) = 1-\frac{5x}{2!}+\frac{(5x)^2}{4!}-\ldots..+\frac{\left(-1\right)^n\left(5x\right)^n}{\left(2n\right)!}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n\left(5x\right)^n}{\left(2n\right)!}$
10	$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\ldots..+\frac{x^n}{n!}+\ldots..=\sum_{n=0}^{\infty}\frac{x^n}{n!}$
11	$e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}\ldots..+\frac{\left(-1\right)^n x^n}{n!}+\ldots..=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^n}{n!}$
12	$\ln\left(1+x\right) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots..+\frac{\left(-1\right)^{n-1}x^n}{n}+\ldots..=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}x^n}{n}$
13	$\ln\left(x\right) = \left(x-1\right)-\frac{\left(x-1\right)^2}{2}+\frac{\left(x-1\right)^3}{3}-\ldots..+\frac{\left(-1\right)^{n-1}\left(x-1\right)^n}{n}+\ldots..=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}\left(x-1\right)^n}{n}$
14	$\frac{1}{x} = 1-\left(x-1\right)+\left(x-1\right)^2-\ldots..+\left(-1\right)^n\left(x-1\right)^n+\ldots..=\sum_{n=0}^{\infty}\left(-1\right)^n\left(x-1\right)^n$
1	$\int x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\ldots.. = \boxed{\mathrm{C}+\frac{x^3}{3}-\frac{x^7}{7\left(3!\right)}+\frac{x^{11}}{11\left(5!\right)}-\ldots..}$
2	$\int_0^1 x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\ldots.. = \frac{x^3}{3}-\frac{x^7}{7\left(3!\right)}+\frac{x^{11}}{11\left(5!\right)}-\ldots..\Big]_0^1=\frac{1}{3}-\frac{1}{42}+\cancel{\frac{1}{1320}}\approx\boxed{0.310}$ $\left[\text{NOTE: }\frac{1}{1320}<0.001\right]$
3	$e^{-0.1}\approx 1-\left(0.1\right)+\frac{\left(0.1\right)^2}{2!}-\frac{\left(0.1\right)^3}{3!}+\frac{\left(0.1\right)^4}{4!}\approx 0.9+\left(0.005\right)-\cancel{\left(-0.00016\right)}=\boxed{0.905}$
4	$1-\left(x/2\right)+\frac{\left(x/2\right)^2}{2!}-\frac{\left(x/2\right)^3}{3!}+\frac{\left(x/2\right)^4}{4!} \Rightarrow \frac{\left(x^4/16\right)}{4!} \Rightarrow \frac{1}{16\left(4!\right)}=\boxed{\frac{1}{384}}$
5	$\int_0^{0.3}x^2e^{-x^2}dx=\int_0^{0.3}x^2\left[1-x^2+\frac{x^4}{2!}-\ldots\right]\Rightarrow\int_0^{0.3}\left(x^2-x^4+\frac{x^6}{2!}-\ldots\right)\Rightarrow\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{14}-\ldots\Big]_0^{0.3}=\boxed{0.009}-\cancel{0.000486}$

TAYLOR SERIES

Taylor: $f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{iv}(a)}{4!}(x-a)^4+.....$

Question #1 (Calculator)

Let f be a function that has derivatives of all orders for all real numbers.
Assume $f(0)=5,\quad f'(0)=-3,\quad f''(0)=1,\quad f'''(0)=4.,\quad f^{iv}(0)=-3,$

1. Write the third-degree Taylor polynomial for f about $x=0$ and use it to approximate $f(0.2)$.
2. Write the fourth-degree Taylor polynomial for g, where $g(x)=f(x^2)$, about $x=0$
3. Write the third-degree Taylor polynomial for h, where $h(x)=\int_0^x f(t)dt$, about $x=0$
4. Let h be defined as in #3. Either find the exact value of $h(1)$ or explain why it can't be determined.

Question #2 (Calculator)

The Taylor series about $x=5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=5$ is given by $f^{(n)}(5)=\frac{(-1)^n n!}{2^n(n+2)}$, and $f(5)=\frac{1}{2}$.

1. Write the third-degree Taylor polynomial for f about $x=5$.
2. Find the radius of convergence of the Taylor series for f about $x=5$
3. Show that the sixth-degree Taylor polynomial for f about $x=5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

Question #3 (No Calculator)

A function f is defined by $f(x)=\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots$ for all x in the interval of convergence of the given power series.

1. Find the **interval of convergence** for this power series. Show the work that leads to your answer.
2. Find $\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x}$
3. Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)dx$.
4. Find the sum of the series determined in #3.

Question #1

1	$P_3(f)(x)=5-3x+\frac{1}{2}x^2+\frac{2}{3}x^3 \Rightarrow f(0.2)\approx P_3(f)(0.2)=5-3(0.2)+\frac{0.04}{2}+\frac{2(0.008)}{3}=\boxed{4.425}$
2	$P_4(g)(x)=P_2(f)(x^2)=\boxed{5-3x^2+\frac{1}{2}x^4}$
3	$P_3(h)(x)=\int_0^x\left(5-3t+\frac{1}{2}t^2\right)dt \Rightarrow \left[5t-\frac{3}{2}t^2+\frac{1}{6}t^3\right]_0^x \Rightarrow \boxed{5x-\frac{3}{2}x^2+\frac{1}{6}x^3}$
4	$h(1)=\int_0^1 f(t)dt$ cannot be determined because $f(t)$ is known only for $\boxed{t=0}$

Question #2

1	$f'(5)=\frac{-1!}{2(3)},\ f''(5)=\frac{2!}{4(4)},\ f'''(5)=\frac{-3!}{8(5)}$ $P_3(f,5)(x)=\boxed{\frac{1}{2}-\frac{1}{6}(x-5)+\frac{1}{16}(x-5)^2-\frac{1}{40}(x-5)^3}$
2	$a_n=\frac{f^{(n)(5)}}{n!}=\frac{(-1)^n}{2^n(n+2)} \Rightarrow \lim_{n\to\infty}\left\lvert\frac{\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\frac{(-1)^n(x-5)^n}{2^n(n+2)}}\right\rvert=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n+2}{n+3}\right)\lvert x-5\rvert=\boxed{\frac{\lvert x-5\rvert}{2}<1}$. The radius of convergence is 2.
3	The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x=6$ is less then the first omitted term in the series. $\boxed{\lvert f(6)-P_6(f,5)(6)\rvert\le\frac{1}{2^7(9)}=\frac{1}{1152}<\frac{1}{1000}}$

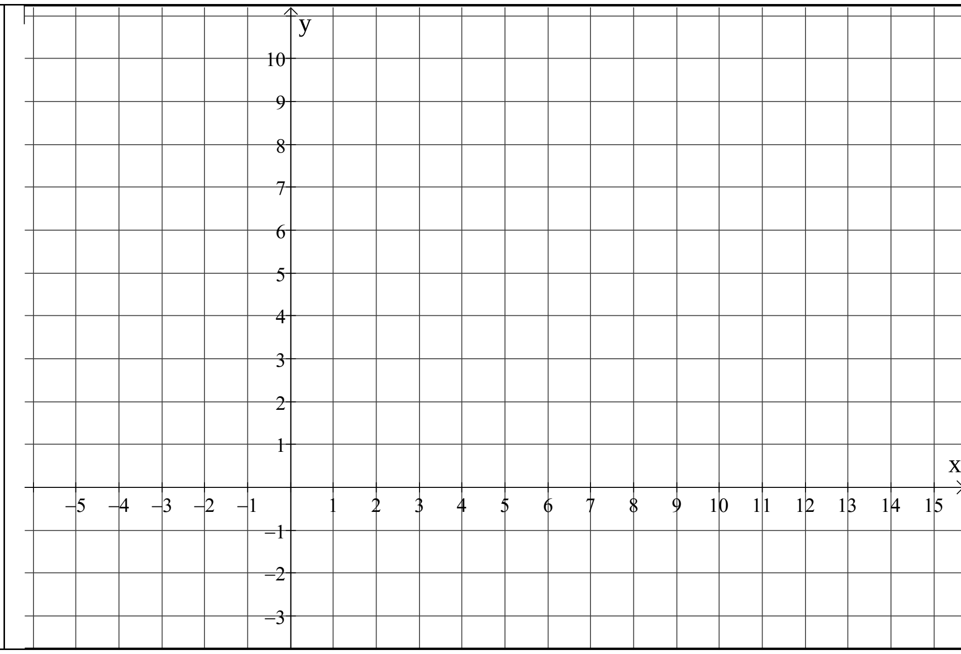
Question #3

1	$\lim_{n\to\infty}\left\lvert\frac{\frac{(n+2)(x)^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}}\right\rvert \Rightarrow \lim_{n\to\infty}\left\lvert\frac{(n+2)}{(n+1)}\frac{x}{3}\right\rvert \Rightarrow \left\lvert\frac{x}{3}\right\rvert<1 \Rightarrow \boxed{-3<x<3}$ At $x=-3$, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3} \Rightarrow$ [Alt. Series] $\Rightarrow \lim_{n\to\infty}\left(\frac{n+1}{3}\right) \Rightarrow$ Diverges At $x=3$, the series is $\sum_{n=0}^{\infty}\frac{n+1}{3} \Rightarrow$ [Divergence Test] Diverges
2	$\lim_{x\to 0}\frac{f(x)-1/3}{x} \Rightarrow \lim_{x\to 0}\left(\frac{2x/9-1/3}{x}+\frac{x^2/9-1/3}{x}+\dots\right)=\lim_{x\to 0}\left(\left(\frac{2}{9}-\frac{1}{3x}\right)+\left(\frac{x}{9}-\frac{1}{3x}\right)+\dots\right)=\boxed{\frac{2}{9}}$
3	$\int_0^1 f(x)=\int_0^1\left(\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots\right)dx=\left[\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\dots+\frac{x^{n+1}}{3^{n+1}}\right]_0^1=\boxed{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\frac{1}{3^{n+1}}+\dots}$
4	$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\frac{1}{3^{n+1}}+\dots \Rightarrow$ [Geometric] $\Rightarrow S_n=\frac{a}{1-r}=\frac{1/3}{1-1/3}=\frac{1/3}{2/3}=\boxed{\frac{1}{2}}$

VOLUME REVIEW

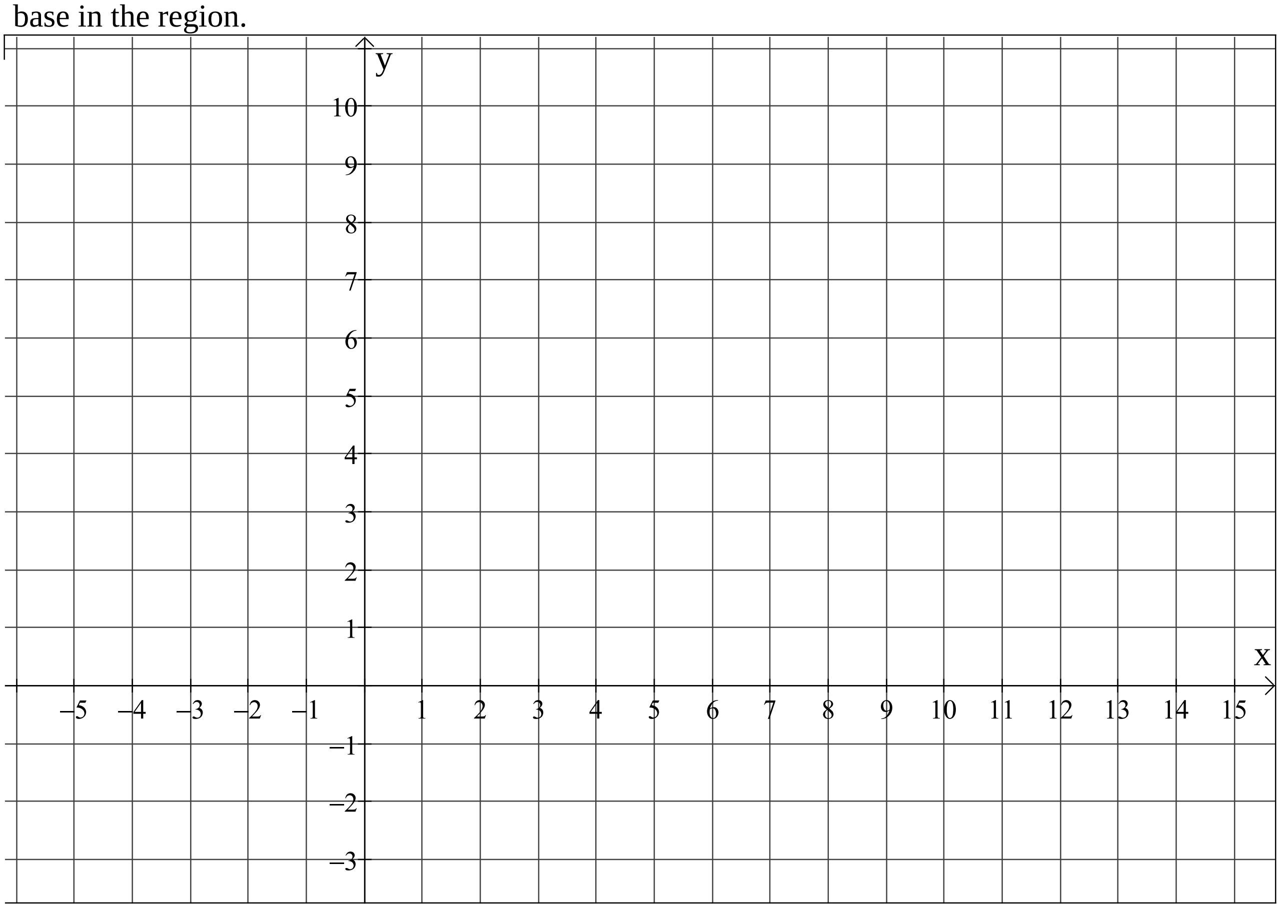
Find the volume of the solid that results when the area of the region enclosed by $y=\sqrt{x}+1$, $x=4$, and $y=1$ is revolved about the ….

1. x−axis
2. y−axis
3. the line $y=1$
4. the line $x=4$
5. the line $y=3$
6. the line $x=-1$
7. the line $y=-1$
8. the line $x=6$
9. the line $y=4$



Find the volume of the solid that results when the area of the region enclosed by $y=\sqrt{x}+1$, $x=4$, and $y=1$

10. has cross sections perpendicular to the x−axis that are squares.
11. has cross sections perpendicular to the x−axis that are semi−circles.
12. has cross sections perpendicular to the x−axis that are rectangles whose height is 5 times the length of its base in the region.



VOLUME REVIEW (ANSWERS)

1.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1\right)^2-(1)^2\right]dx=\boxed{58.643}$	(Washer)
2.	$V=\pi\int_1^3\left[(4)^2-\left((y-1)^2\right)^2\right]dy=\boxed{80.425}$	(Washer)
3.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1-1\right)^2\right]dx=\boxed{25.133}$	(Disk)
4.	$V=\pi\int_1^3\left[\left(4-(y-1)^2\right)^2\right]dy=\boxed{53.617}$	(Disk)
5.	$V=\pi\int_0^4\left[(3-1)^2-\left(3-(\sqrt{x}+1)\right)^2\right]dx=\boxed{41.888}$	(Washer)
6.	$V=\pi\int_1^3\left[\left(4-(-1)\right)^2-\left((y-1)^2-(-1)\right)^2\right]dy=\boxed{113.935}$	(Washer)
7.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1-(-1)\right)^2-\left(1-(-1)\right)^2\right]dx=\boxed{92.153}$	(Washer)
8.	$V=\pi\int_1^3\left[\left(6-(y-1)^2\right)^2-(6-4)^2\right]dy=\boxed{120.637}$	(Washer)
	$V=2\pi\int_0^4\left[(6-x)\left(\sqrt{x}\right)\right]dx=\boxed{120.637}$	(Shell)
9.	$V=\pi\int_0^4\left[(4-1)^2-\left(4-(\sqrt{x}+1)\right)^2\right]dx=\boxed{75.398}$	(Washer)
10.	$V=\int_0^4(\sqrt{x}+1-1)^2\,dx=\boxed{8}$	$\{\text{Area}=s^2\}$
11.	$V=\frac{1}{2}\pi\int_0^4\left(\frac{\sqrt{x}}{2}\right)^2dx=\boxed{3.142}$	$\{\text{Area}=\frac{1}{2}\pi r^2\}$
12.	$V=\int_0^4\left[\left(\sqrt{x}\right)\left(5\sqrt{x}\right)\right]dx=\boxed{40}$	$\{\text{Area}=5s^2\}$

Reminders: If calculator, make sure to go to **AT LEAST** three decimal places.

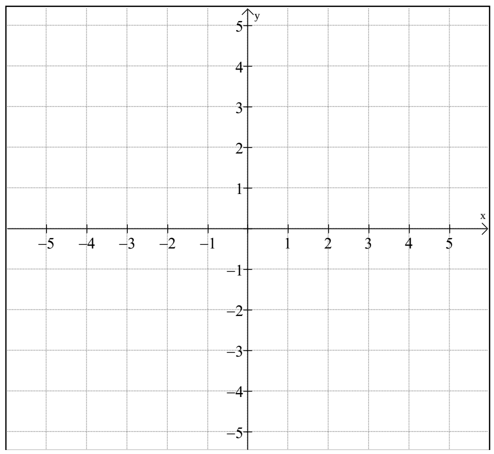
This will most likely be a **NON-CALCULATOR** problem this year.

POLAR EQUATIONS

<u>Question #1 (Calculator)</u>

Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$

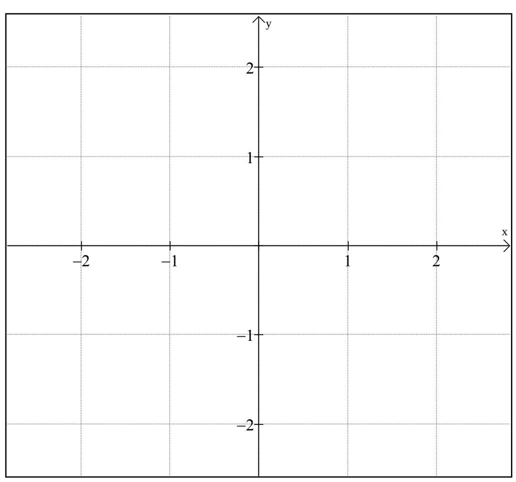
1. Sketch the two polar curves in the *xy*–plane provided below and shade the region R.
2. Find the intersection (θ) of the two curves.
3. Find the area of R.
4. Find the area of $r = 2(1 - \sin\theta)$.
5. Find the TOTAL AREA of the object.



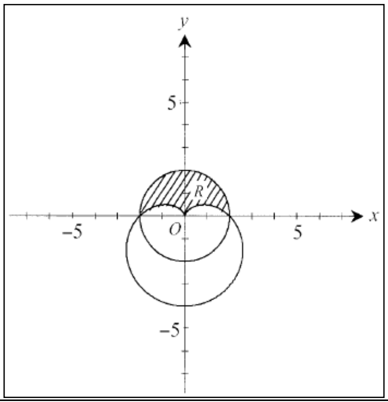
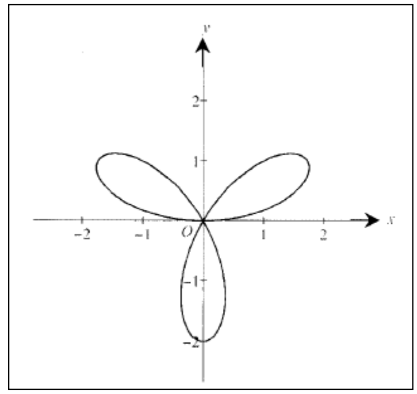
<u>Question #2 (No Calculator)</u>

Consider the polar curve $r = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

1. In the *xy*–plane provided below, sketch the curve.
2 Find the area of the region inside the curve.
3. Find the slope of the curve at the point where $\theta = \dfrac{\pi}{4}$



POLAR EQUATIONS
(ANSWERS)

GRAPH FOR PROBLEM #1	GRAPH FOR PROBLEM #2
	

QUESTION #1

2	$2=2(1-\sin\theta) \;\Rightarrow\; 1-\sin\theta=1 \;\Rightarrow\; \sin\theta=0 \;\Rightarrow\; \boxed{\theta=0 \text{ or } \pi}$
3	$\text{Area}=\frac{1}{2}\int_0^{\pi} r^2 d\theta \;\Rightarrow\; \frac{1}{2}\int_0^{\pi}\left[2^2-\left(2(1-\sin\theta)\right)^2\right]d\theta \;\Rightarrow\; \frac{1}{2}\int_0^{\pi}\left[4-4(1-\sin\theta)^2\right]d\theta \;\Rightarrow$ $2\int_0^{\pi}\left[1-(1-\sin\theta)^2\right]d\theta \;\Rightarrow\; 2\int_0^{\pi}\left[2\sin\theta-\sin^2\theta\right]d\theta \;\Rightarrow\; 4\int_0^{\pi}\sin\theta d\theta-2\left(\frac{1}{2}\right)\int_0^{\pi}(1-\cos 2\theta)d\theta \;\Rightarrow$ $-4\cos\theta\Big\|_0^{\pi}-\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi}=\left[-4(-1)+4(1)\right]-\left[\pi-0\right]=\boxed{8-\pi}$
4	$\text{Area}=\frac{1}{2}\int_0^{2\pi} r^2 d\theta \;\Rightarrow\; \frac{1}{2}\int_0^{2\pi}\left(2(1-\sin\theta)\right)^2 d\theta \;\Rightarrow\; 2\int_0^{2\pi}\left(1-2\sin\theta+\sin^2\theta\right)d\theta \;\Rightarrow$ $2\int_0^{2\pi}\left(1-2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta=2\left[\frac{3\theta}{2}+2\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}=2\left[(3\pi+2)-(2)\right]=\boxed{6\pi}$
5	$8-\pi+6\pi=\boxed{8+5\pi}$

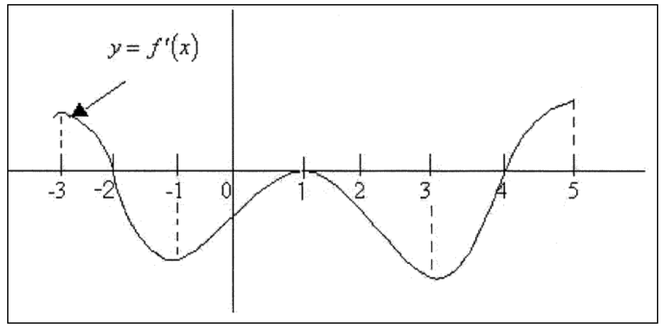
QUESTION #2

2	$\text{Area}=\frac{1}{2}\int_0^{\pi}4\sin^2 3\theta d\theta \;\Rightarrow\; 2\int_0^{\pi}\frac{1}{2}(1-\cos 6\theta)d\theta \;\Rightarrow\; \int_0^{\pi}(1-\cos 6\theta)d\theta \;\Rightarrow\; \left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\pi}=\boxed{\pi}$ OR: $\text{Area}=3\bullet\frac{1}{2}\int_0^{\pi/3}4\sin^2 3\theta d\theta \;\Rightarrow\; 6\int_0^{\pi/3}\frac{1}{2}(1-\cos 6\theta)d\theta \;\Rightarrow\; 3\left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\pi/3}=3\left[\frac{\pi}{3}-0\right]_0^{\pi/3}=\boxed{\pi}$
3	$x=r\cos\theta \;\Rightarrow\; x=2\sin 3\theta\cos\theta \;\Rightarrow\; \frac{dx}{d\theta}=6\cos 3\theta\cos\theta+2\sin 3\theta(-\sin\theta) \;\Rightarrow$ $\frac{dx}{d\theta}\Big\|_{\theta=\frac{\pi}{4}}=6\left(\cos\frac{3\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)+2\left(\sin\frac{3\pi}{4}\right)\left(-\sin\frac{\pi}{4}\right)=\left(-3\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right)+2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)=\boxed{-4}$ $y=r\sin\theta \;\Rightarrow\; y=2\sin 3\theta\sin\theta \;\Rightarrow\; \frac{dy}{d\theta}=6\cos 3\theta\sin\theta+2\sin 3\theta(\cos\theta) \;\Rightarrow$ $\frac{dy}{d\theta}\Big\|_{\theta=\frac{\pi}{4}}=6\left(\cos\frac{3\pi}{4}\right)\left(\sin\frac{\pi}{4}\right)+2\left(\sin\frac{3\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)=6\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)=\boxed{-2} \quad \therefore \frac{dy}{dx}=\frac{-2}{-4}=\boxed{\frac{1}{2}}$

Analyzing the Graph of a Derivative:

PROBLEM #1

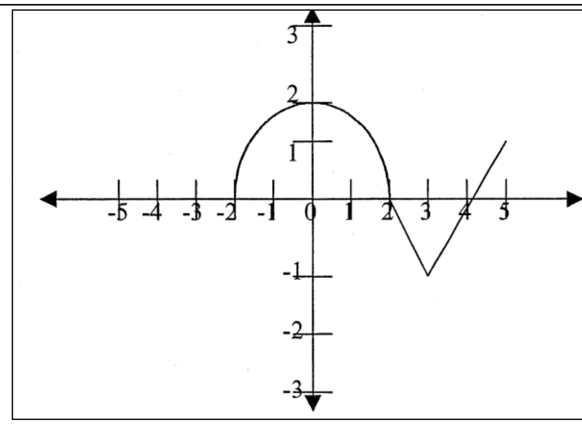
1. For what value(s) of x does *f* have a relative maximum? Why?
2. For what value(s) of x does *f* have a relative minimum? Why?
3. On what intervals is the graph of *f* concave up? Why?
4. On what intervals is *f* increasing? Why?
5. For what value(s) of x does *f* have an inflection point? Why?



PROBLEM #2

The graph of a function *f* consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$

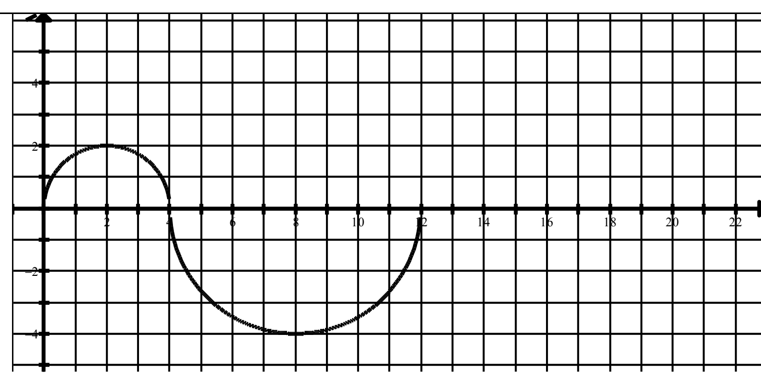
1. Find g(3).
2. For what value(s) of x does *g* have a relative maximum? Why?
3. For what value(s) of x does *g* have a relative minimum? Why?
4. For what value(s) of x does g have an inflection point? Why?
5. Write an equation for the line tangent to the graph of *g* at *x*=3



PROBLEM #3

The graph below shows f', the derivative of function f. The graph consists of two semi−circles and one line segment. Horizontal tangents are located at x = 2 and x = 8 and a vertical tangent is located at x = 4.

1. On what intervals is f increasing? Justify your answer.
2. For what values of x does f have a relative minimum? Justify.
3. On what intervals is f concave up? Justify.
4. For what values of x is f'' undefined?
5. Identify the x−coordinates for all points of inflection of f.
6. For what value of x does f reach its maximum value? Justify.
7. If f (4) = 5, find f(12).



PROBLEM #1 ANSWERS:

1. at $x=-2$, because $f'(x)$ changes from positive to negative at $x=-2$.
2. at $x=4$, because $f'(x)$ changes from negative to positive at $x=4$.
3. (−1,1) and (3,5) because $f'(x)$ is increasing on these intervals (thus $f''(x)>0$).
4. (−3,−2) and (4,5) because $f'(x)>0$ on these intervals.
5. at x = −1, x = 1 and x = 3, because $f''(x)$ changes signs at these values of x.

PROBLEM #2 ANSWERS:

1. $g(3)=\int_0^3 f(t)\,dt=\left(\frac{1}{4}\right)(\pi)\left(2^2\right)+\left(\frac{1}{2}\right)(1)(-1)=\pi-\frac{1}{2}$
2. At $x=2$, because $g'(x)=f(x)$ changes from positive to negative at $x=2$.
3. At $x=4$, because $g'(x)=f(x)$ changes from negative to positive at $x=4$.
4. At $x=0$ and $x=3$, because $g''(x)=f'(x)$ changes from positive to negative at $x=0$ and changes from negative to positive at $x=3$.
5. $g(3)=\pi-\frac{1}{2}$ and $g'(3)=f(3)=-1$ $\therefore$ $y-\left(\pi-\frac{1}{2}\right)=-1(x-3)$

PROBLEM #3 ANSWERS:

1. (0,4) and (12,22) because $f'(x)>0$ for these values of x.
2. At $x=12$ because $f'(x)$ changes from negative to positive at $x=12$.
3. (0,2) (8,12) and (12,22) because $f''(x)>0$ for these values of x. [Or, $f'(x)$ is increasing for these values of x.]
4. At $x=4$ and $x=12$
5. At $x=2$ and $x=8$
6. At $x=4$ because $f(x)$ is increasing on (0,4) [$f'(x)>0$] and $f(x)$ is decreasing on (4,12) [$f'(x)<0$] and $f(x)$ is increasing on (12,22) [$f'(x)>0$]. Thus, max can occur at $x=4$ or at $x=22$ On the interval (4,12), $f(x)$ decreases by: $\int_4^{12} f'(x)\,dx=8\pi$ On the interval (12,22), $f(x)$ increases by: $\int_{12}^{22} f'(x)\,dx=25$ Because $8\pi>25$, $f(x)$ decreases by $8\pi-25$ on the interval (4,22). Thus, maximum occurs at $x=4$.
7. $f(12)-f(4)=\int_4^{12} f'(x)\,dx \Rightarrow f(12)=f(4)+\int_4^{12} f'(x)\,dx = 5-8\pi$

PARAMETRIC EQUATIONS

<u>Question #1 (Calculator)</u>

An object is moving along a curve in the *xy*–plane has position $\left(x(t), y(t)\right)$ at time *t* with:
$\frac{dx}{dt} = \cos\left(t^3\right)$ and $\frac{dy}{dt} = 3\sin\left(t^2\right)$ for $0 \le t \le 3$. At time $t = 2$, the object is at position $(4,5)$.

1. Write an equation for the line tangent to the curve at $(4,5)$.
2. Find the speed of the object at time $t = 2$.
3. Find the total distance traveled by the object over the time interval $0 \le t \le 1$.
4. Find the position of the object at time $t = 3$

<u>Question #2 (No Calculator)</u>

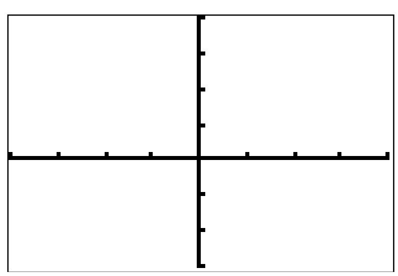
A moving particle has position $\left(x(t), y(t)\right)$ at time *t*. The position of the particle at time $t = 1$ is $(2,6)$ and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

1. Find the acceleration vector at time $t = 3$.
2 Find the position of the particle at time $t = 3$.
3. For what time $t > 0$ does the line tangent to the path of the particle at $\left(x(t), y(t)\right)$ have a slope of 8?
4. The particle approaches a line at $t \to \infty$. Find the slope of this line. Show work that leads to conclusion.

<u>Question #3 (Calculator)</u>

A particle moves in the *xy*–plane so that its position at any time *t*, $0 \le t \le \pi$, is given by:
$x(t) = \frac{t^2}{2} - \ln(1+t)$ and $y(t) = 3\sin t$

1. Sketch the path of the particle in the *xy*–plane below. Indicate the direction of motion along the path.



2. At what time *t*, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $\left(x(t), y(t)\right)$ of the particle at this time?

3. At what time *t*, $0 \le t \le \pi$, is the particle on the y–axis? Find the speed and the acceleration vector of the particle at this time.

PARAMETRIC EQUATIONS (ANSWERS)

Question #1

1

$$\frac{dy}{dx}=\frac{3\sin\left(t^2\right)}{\cos\left(t^3\right)} \Rightarrow \left.\frac{dy}{dx}\right\rvert_{t=2}=\frac{3\sin 4}{\cos 8}\approx \boxed{15.604} \Rightarrow \boxed{\boxed{y-5=15.604(x-4)}}$$

2/3

$$\text{Speed} = \sqrt{\cos^2(8)+9\sin^2(4)}\approx \boxed{\boxed{2.275}} \quad \text{AND} \quad \text{Distance} = \int_0^1 \sqrt{\cos^2\left(t^3\right)+9\sin^2\left(t^2\right)}\,dt \approx \boxed{\boxed{1.458}}$$

4

$$x(3)=4+\int_2^3 \cos\left(t^3\right)\approx \boxed{\boxed{3.954}} \quad \text{and} \quad y(3)=5+\int_2^3 3\sin\left(t^2\right)\approx \boxed{\boxed{4.906}}$$

Question #2

1

$$\left(x''(t),y''(t)\right)=\left(\frac{2}{t^3},\frac{-2}{t^3}\right) \Rightarrow \left(x''(3),y''(3)\right)=\boxed{\boxed{\left(\frac{2}{27},\frac{-2}{27}\right)}}$$

2

$$\left(x(t),y(t)\right)=\left(t+\frac{1}{t}+C_1,\ 2t-\frac{1}{t}+C_2\right) \Rightarrow (2,6)=\left(x(1),y(1)\right)=\left(2+C_1,1+C_2\right) \Rightarrow C_1=0,\ \ C_2=5$$

$$x(3)=x(1)+\int_1^3 1-\frac{1}{t^2}\,dt=2+\left[t+t^{-1}\right]_1^3=2+\left(3+\frac{1}{3}\right)-(1+1)=\boxed{\boxed{\frac{10}{3}}} \ \& \ y(3)=6+\left(6-\frac{1}{3}\right)-(2-1)=\boxed{\boxed{\frac{32}{3}}}$$

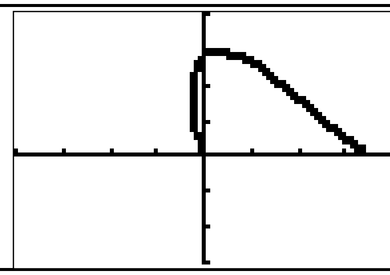
3/4

$$\frac{dy}{dx}=\frac{2+\frac{1}{t^2}}{1-\frac{1}{t^2}}=8 \Rightarrow 2+\frac{1}{t^2}=8\left(1-\frac{1}{t^2}\right) \Rightarrow \frac{9}{t^2}=6 \Rightarrow t^2=\frac{9}{6} \Rightarrow \boxed{\boxed{t=\sqrt{\frac{3}{2}}}} \qquad \lim_{t\to\infty}\frac{dy}{dx}=\lim_{t\to\infty}\left(\frac{2+\frac{1}{t^2}}{1-\frac{1}{t^2}}\right)=\boxed{\boxed{2}}$$

Question #3

1

Make sure to indicate direction by drawing arrows.
Direction is clockwise.



2

$$x'(t)=t-\frac{1}{1+t}=0 \Rightarrow t^2+t-1=0 \Rightarrow t=\frac{-1+\sqrt{5}}{2}\approx \boxed{\boxed{0.618}}$$

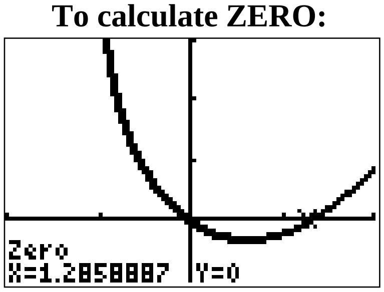
$$x(0.618)\approx \boxed{\boxed{-0.290}} \quad \text{and} \quad y(0.618)\approx \boxed{\boxed{1.738}}$$

3

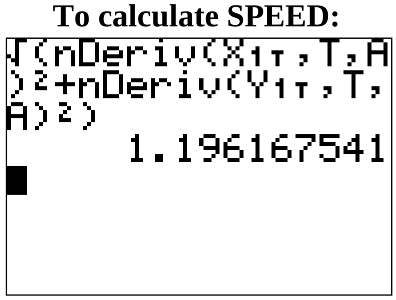
$$x(t)=\frac{t^2}{2}-\ln(1+t)=0 \Rightarrow \boxed{\boxed{t\approx 1.286}} \quad \text{AND} \quad \text{Speed}=\sqrt{x'(1.286)^2+y'(1.286)^2}\approx \boxed{\boxed{1.196}}$$

$$\text{Acceleration}=\left\langle x''(1.286),y''(1.286)\right\rangle \approx \boxed{\boxed{\langle 1.191,-2.879\rangle}}$$

To calculate ZERO:



To calculate SPEED:



To calculate ACCELERATION:

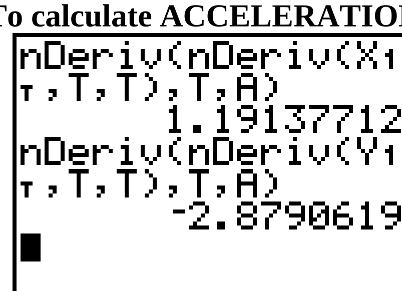


TABLE OF VALUES (Calculator)
(THE WIRE)

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ $(^\circ\text{C})$	100	89	73	64	51

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius $(^\circ\text{C})$, of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

1. Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

2. Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

3. Find $\int_0^8 T'(x)\,dx$ and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

4. Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

TABLE OF VALUES (ANSWERS)
(THE WIRE)

1.	$\frac{T(8)-T(6)}{8-6}=\frac{51-64}{2}=\frac{-13}{2}$ °C/cm
2.	$\frac{1}{8}\int_0^8 T(x)\,dx=\frac{1}{8}\left[\frac{100+89}{2}(1)+\frac{89+73}{2}(4)+\frac{73+64}{2}(1)+\frac{64+51}{2}(2)\right]\approx 75.25$ °C
3.	$\int_0^8 T'(x)\,dx=T(8)-T(0)=51-100=-49$ °C The temperature drops 49 °C from the heated and of the wire to the other end of the wire.
4.	No. The MVT guarantees that $T''(x)<0$ for some value of x on the interval $1<x<6$ (See graph of $T'(x)$ below)

4.

x	$T(x)$	$T'(x)$ (estimate)
1	89	$\frac{89-100}{1-0}=-11$
5	73	$\frac{73-89}{5-1}=-4$
6	64	$\frac{64-73}{6-5}=-9$
8	51	$\frac{51-64}{8-6}=-6.5$

Graph of $T'(x)$

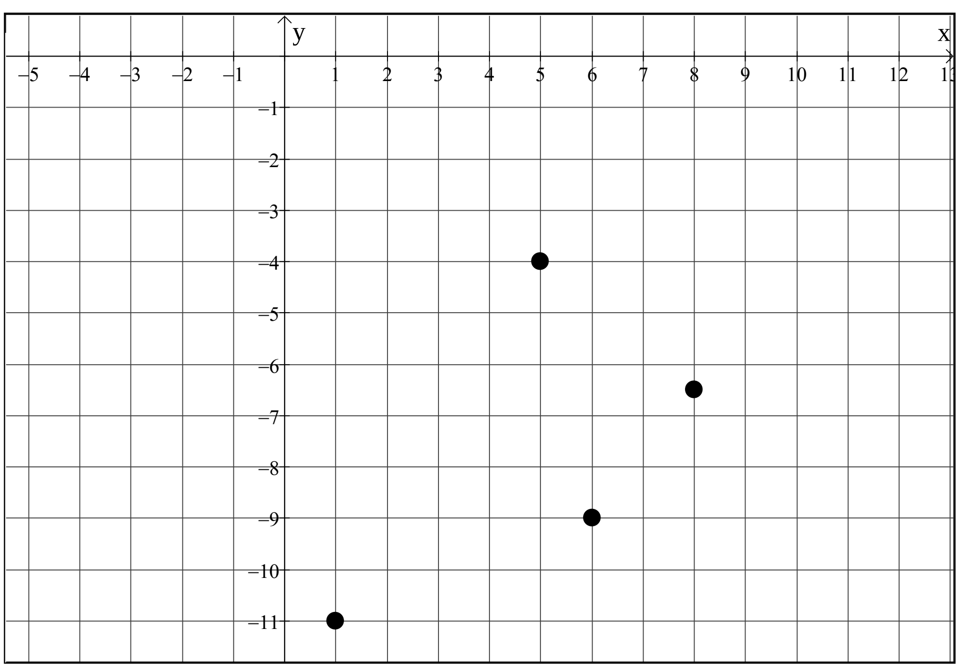


TABLE OF VALUES (Calculator)
(WATER TEMPERATURE)

t (days)	$W(t)$ (°C)
0	20
3	25
6	28
9	27
12	22
15	19

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

1) Use data from the table to find the average change in the water temperature for the 15-day period.

2) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

3) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

4) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

5) Use the function P defined in part (4) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

6) Will $W'(t) = 0$ during the 15-day period? Why or why not?

TABLE OF VALUES (Answers)
(WATER TEMPERATURE)

1.	$\frac{W(15)-W(0)}{15-0}=\frac{19-20}{15}=\boxed{\frac{-1}{15}\ \frac{°\text{C}}{\text{day}}}$
2.	$\frac{W(15)-W(9)}{15-9}=\frac{19-27}{6}=\boxed{\frac{-4}{3}\ \frac{°\text{C}}{\text{day}}}$ or: $\frac{W(15)-W(12)}{15-12}=\frac{19-22}{3}=\boxed{-1\ \frac{°\text{C}}{\text{day}}}$ or: $\frac{W(12)-W(9)}{12-19}=\frac{22-27}{3}=\boxed{\frac{-5}{3}\ \frac{°\text{C}}{\text{day}}}$
3.	$\frac{1}{15}\bullet\frac{3}{2}\left[20+2(25)+2(28)+2(27)+2(22)+19\right]=\boxed{24.3\ °\text{C}}$
4.	$P'(t)=10e^{(-t/3)}+10te^{(-t/3)}\left(\frac{-1}{3}\right)\Rightarrow P'(12)=10e^{(-4)}-40e^{(-4)}\ =\boxed{\frac{-30}{e^4}\ \frac{°\text{C}}{\text{day}}}$ Remember, Calc Derivative: $P'(12)\approx\boxed{-0.549}$ or Math 8, y_1, x, $12\approx\boxed{-0.549}$ This means that the temperature is decreasing at the rate of $30e^{-4}\ \frac{°\text{C}}{\text{day}}$ at $t=12$ days
5.	$\frac{1}{15}\int_0^{15}20+10te^{(-t/3)}dt\approx\boxed{25.757\ °\text{C}}$
6.	Yes. By Mean Value Theorem, $W(0)=20$ and somewhere $12<\text{t}<15$, $W(t)=20$ Thus, $W'(t)=0$ somewhere $0<\text{t}<15$

TABLE OF VALUES (Calculator)
(PIE PROBLEM)

1. Let $y(t)$ represent the temperature of a pie that has been removed from a 450°F oven and left to cool in a room with a temperature of 72°F, where y is a differentiable function of t. The table below shows the temperature recorded every five minutes.

t (min)	0	5	10	15	20	25	30
$y(t)$ $(^\circ\text{F})$	450	388	338	292	257	226	200

A) Use data from the table to find an approximation for $y'(18)$, and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Show the computations that lead to your answer, and indicate units of measure.

B) Use data from the table to find the value of $\int_{10}^{25} y'(t)\,dt$, and explain the meaning of $\int_{10}^{25} y'(t)\,dt$ in terms of the temperature of the pie. Indicate units of measure.

C) A model for the temperature of the pie is given by the function: $W(t)=72+380e^{-0.036t}$ where t is measured in minutes and $W(t)$ is measured in degrees Fahrenheit $(^\circ\text{F})$. Use the model to find the value of $W'(18)$. Indicate units of measure.

D) Use the model given in part (c) to find the time at which the temperature of the pie is 300°F.

TABLE OF VALUES (Answers)
(PIE PROBLEM)

A) Use data from the table to find an approximation for $y'(18)$, and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Show the computations that lead to your answer, and indicate units of measure.

$$y'(18) \approx \frac{257-292}{20-15} = \boxed{-7^{\circ}\ \frac{\text{F}}{\text{min.}}}$$

When $t = 18$ minutes, the temperature of the pie is decreasing at a rate of approximately 7° F per minute.

B) Use data from the table to find the value of $\int_{10}^{25} y'(t)\,dt$, and explain the meaning of $\int_{10}^{25} y'(t)\,dt$ in terms of the temperature of the pie. Indicate units of measure.

$$\int_{10}^{25} y'(t)\,dt = y(25) - y(10) = \boxed{-112^{\circ}\text{F}}$$

From $t = 10$ minutes to $t = 25$ minutes, the temperature of the pie dropped 112° F.

C) A model for the temperature of the pie is given by the function: $W(t) = 72 + 380e^{-0.036t}$ where t is measure in minutes and $W(t)$ is measured in degrees Fahrenheit $\left({}^{\circ}\text{F}\right)$. Use the model to find the value of $W'(18)$. Indicate units of measure.

$$W'(18) = \boxed{-7.156^{\circ}\text{F}} \text{ per minute.}$$

D) Use the model given in part (c) to find the time at which the temperature of the pie is 300° F.

$$W(t) = 300 \text{ when } t = \boxed{14.190 \text{ minutes}}.$$

TABLE OF VALUES
(SUGAR MILL)

2. Let $y(t)$ represent the population of the town of Sugar Mill over a 10-year period, where y is a differentiable function of t. The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
$y(\text{people})$	2500	2912	3360	3815	4330	4875

A) Use data from the table to find an approximation for $y'(7)$, and explain the meaning of $y'(7)$ in terms of the population of Sugar Mill. Show the computations that lead to your answer.

B) Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \le t \le 10$ by using a left Reimann sum with five equal subintervals. Show the computations that lead to your answer.

C) A model for the population of another town, Pine Grove, over the same 10-year period is given by the function $P(t) = (2t+50)^2$, where t is measured in years and P(t) is measured in people. Use the model to find the value of P'(7).

D) Use the model given in part (c) to find the value of $\frac{1}{10}\int_0^{10} P(t)\,dt$. Explain the meaning of this integral expression in terms of the population of Pine Grove.

TABLE OF VALUES (Answers)
(SUGAR MILL)

2. Let y(t) represent the population of the town of Sugar Mill over a 10-year period, where y is a differentiable function of t. The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
$y(\text{people})$	2500	2912	3360	3815	4330	4875

A) Use data from the table to find an approximation for y'(7), and explain the meaning of y'(7) in terms of the population of Sugar Mill. Show the computations that lead to your answer.

$$y'(7) = \frac{4330-3815}{8-6} = \boxed{257.5}$$

When $t = 7$ years, the population of Sugar Mill is increasing at a rate of approximately 257.5 people per year.

B) Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \le t \le 10$ By using a left Reimann sum with five equal subintervals. Show the computations that lead to your answer.

$$\text{Average Population} = \frac{1}{10}\int_0^{10} y(t)\,dt = \frac{1}{10}\left(2\left(2500+2912+3360+3815+4330\right)\right) = \boxed{3383.4}$$

so the average population over the 10 – year period was approximately 3383.4 people.

C) A model for the population of another town, Pine Grove, over the same 10-year period is given by the function $P(t) = (2t+50)^2$, where t is measured in years and P(t) is measured in people. Use the model to find the value of P'(7).

$$\boxed{P'(7) = 256 \text{ people per year}}$$

D) Use the model given in part (c) to find the value of $\frac{1}{10}\int_0^{10} P(t)dt$. Explain the meaning of this integral expression in terms of the population of Pine Grove.

$\frac{1}{10}\int_0^{10} P(t)dt = \boxed{3633.333 \text{ people}}$. This means that the average population of Pine Grove over the 10 – year period was approximately 3633.333 people.

TABLE OF VALUES
(BOWL OF SOUP)

3. A bowl of soup is place on the kitchen counter to cool. Let $T(x)$ represent the temperature of the soup at time x, where T is a differentiable function of x. The temperature of the soup at selected times is given in the table below.

x (min)	0	4	7	12
$T(x)\left({}^\circ F\right)$	108	101	99	95

A) Use data from the table to find:

$$\int_0^{12} T'(x)\,dx$$

Explain the meaning of this definite integral in terms of the temperature of the soup.

B) Use data from the table to find the average rate of change of $T(x)$ over the time interval $x = 4$ to $x = 7$

C) Explain the meaning of:

$$\frac{1}{12}\int_0^{12} T(x)\,dx$$

In terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals

TABLE OF VALUES (ANSWERS)
(BOWL OF SOUP)

3. A bowl of soup is place on the kitchen counter to cool. Let T(x) represent the temperature of the soup at time x, where T is a differentiable function of x. The temperature of the soup at selected times is given in the table below.

x (min)	0	4	7	12
$T(x)\left({}^{\circ}F\right)$	108	101	99	95

A) Use data from the table to find:

$$\int_0^{12} T'(x)\,dx$$

Explain the meaning of this definite integral in terms of the temperature of the soup.

$$\int_0^{12} T'(x)\,dx = T(12) - T(6) = \boxed{-13^{\circ}F}$$

From $x = 0$ to $x = 12$ minutes, the temperature of the soup dropped $13^{\circ}F$

B) Use data from the table to find the average rate of change of $T(x)$ over the time interval $x = 4$ to $x = 7$

Average rate of change = $\dfrac{T(7)-T(4)}{7-4} = \dfrac{99-101}{3} = -\dfrac{2}{3}^{\circ} F/\text{min}$

C) Explain the meaning of:

$$\frac{1}{12}\int_0^{12} T(x)\,dx$$

In terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals

$\dfrac{1}{12}\displaystyle\int_0^{12} T(x)\,dx$ represents the average temperature of the soup over the 12-minute period

and is approximately equal to:

$$\frac{1}{12}\left(\frac{1}{2}(4)(108+101)+\frac{1}{2}(3)(101+99)+\frac{1}{2}(5)(99+95)\right) = \boxed{100.25^{\circ}F}$$

TABLE OF VALUES
(WATER INTO A TANK)

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function $R(t)$. A table of values of $R(t)$, for the time interval $0 \le t \le 20$ minutes, is shown below

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

A) Use a right Riemann sum with four subintervals to approximate the value of:

$$\int_0^{20} R(t)\,dt$$

Is your approximation greater or less than the true value? Give a reason for your answer.

B) A model for the rate at which water is being pumped into the tank is given by the function:

$$W(t) = 25e^{0.03t}$$

where t is measured in minutes and $W(t)$ is measured in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from $t = 0$ to $t = 20$ minutes.

C) The tank contained 100 gallons of water at time $t = 0$. Use the model given in part (b) to find the amount of water in the tank at t = 20 minutes

TABLE OF VALUES (ANSWERS)
(WATER INTO A TANK)

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function $R(t)$. A table of values of $R(t)$, for the interval $0 \le t \le 20$ minutes, is shown below

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

A) Use a right Riemann sum with four subintervals to approximate the value of:

$$\int_0^{20} R(t)\,dt$$

Is your approximation greater or less than the true value? Give a reason for your answer.

$$\int_0^{20} R(t)\,dt \approx (4)(28)+(5)(33)+(8)(42)+(3)(46) = \boxed{751 \text{ gallons}}$$

Since R is positive, this is an estimate of the amount of water pumped into the tank during the 20-minute period. Since R increases on $0 < t < 20$, the right Riemann sum approximation of 751 gallons is greater than $\int_0^{20} R(t)\ dt$.

B) A model for the rate at which water is being pumped into the tank is given by the function:

$$W(t) = 25e^{0.03t}$$

Where t is measured in minutes and $W(t)$ is measured in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from $t = 0$ to $t = 20$ minutes.

$$\text{Average Rate} = \frac{1}{20}\int_0^{20} W(t)\,dt = \boxed{34.255 \text{ gal/min}}$$

C) The tank contained 100 gallons of water at time $t = 0$. Use the model given in part (b) to find the amount of water in the tank at t = 20 minutes

$$100 + \int_0^{20} W(t)\,dt = \boxed{785.099 \text{ gallons}}$$

TABLE OF VALUES
(CAR VELOCITY)

5. Car A has positive velocity $v_A(t)$ as it travels on a straight road, where v_A is a differentiable function of t. The velocity is recorded for selected values over the time interval $0 \le t \le 10$ seconds, as shown in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (ft/sec)	1	9	36	61	115

A) Use data from the table to approximate the acceleration of Car A at t = 8 seconds. Indicate units of measure.

B) Use data from the table to approximate the distance traveled by Car A over the interval $0 \le t \le 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

C) Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2 \quad ft/\sec^2$. At time t = 3 seconds, the velocity of car B is 11 ft/sec. Which car is traveling faster at time t = 7 seconds? Explain your answer.

TABLE OF VALUES (Answers)
(CAR VELOCITY)

5. Car A has positive velocity $v_A(t)$ as it travels on a straight road, where v_A is a differentiable function of t. The velocity is recorded for selected values over the time interval $0 \le t \le 10$ seconds, as shown in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (ft/sec)	0	9	36	61	115

A) Use data from the table to approximate the acceleration of Car A at t = 8 seconds. Indicate units of measure.

Let $a_A(t)$ be the acceleration of Car A at time t. Then:

$$a_A(8) \approx \frac{v_A(10) - v_A(7)}{10-7} = \frac{115-61}{3} = 18\,\frac{ft/\text{sec}}{\text{sec}} = \boxed{18\,\frac{ft}{\text{sec}^2}}$$

B) Use data from the table to approximate the distance traveled by Car A over the interval $0 \le t \le 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

Distance: $$\int_0^{10} v_A\,dt \approx \frac{1}{2}\left[(2)(0+9)+(3)(9+36)+(2)(36+61)+(3)(61+115)\right] = \boxed{437.5\,ft}$$

C) Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2\,ft/\text{sec}^2$. At time t = 3 seconds, the velocity of car B is 11 ft/sec. Which car is traveling faster at time t = 7 seconds? Explain your answer.

Let be the velocity of Car B at time t. Then: $v_B(t) = \int (2t+2)\,dt = t^2 + 2t + \text{C}$

At $t = 3$, we have $11 = 9 + 6 + \text{C}$, so that $\text{C} = -4$ and $\boxed{v_B(t) = t^2 + 2t - 4}$.

Hence, $v_B(7) = 59 < 61 = v_A(7)$. $\boxed{\text{We conclude that Car A is traveling faster at time } t = 7 \text{ seconds.}}$

RATES OF CHANGE

PROBLEM #1 (CALCULATOR)

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by:

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \quad \text{for } 0 \le t \le 10$$

Where $F(t)$ is measured in cars per minute and t is measured in minutes.

1. To the nearest whole number, how many cars pass through the intersection over the 10−minute period?

2. Is the traffic flow increasing or decreasing at $t = 5$? Justify.

3. What is the average value of the traffic flow over the time interval $3 \le t \le 7$? Indicate units of measure.

4. What is the average rate of change of the traffic flow over the time interval $3 \le t \le 7$? Indicate units of measure.

5. At what time, t, is the traffic flow the greatest? What is the greatest flow?

PROBLEM #2 (CALCULATOR)

A water tank at Camp Diamond Bar holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the tank at the rate:

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

---

1. Is the amount of water in the tank increasing at time $t = 5$? Why or why not?

2. To the nearest whole number, how many gallons of water are in the tank at time $t = 12$?

3. At what time, t, for $0 \le t \le 12$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

4. For $t > 12$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank become empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

5. What is the average rate of change in the amount of water in tank for $0 \le t \le 12$ hours?

PROBLEM #1 ANSWERS:

1. $\int_0^{10} F(t)dt \approx 826$ cars
2. Decreasing. Because $F'(5) < 0$ $\left[F'(5) \approx -1.602\right]$
3. $\dfrac{1}{7-3}\int_3^7 F(t)\,dt = 84.014$ cars/minute
4. $\dfrac{1}{7-3}\int_3^7 F'(t)dt = \dfrac{F(7)-F(3)}{7-3} = -1.348$ cars/minute^2
5. F(0) = 82, F(12) = 78.164, and F(π) = 86 – so greatest at $t = \pi$ minutes, 86 cars per minute.

PROBLEM #2 ANSWERS:

1. No. the amount of water is not increasing at $t = 5$ because $W(5) - R(5) = -156.0998 < 0$
2. 1643 gallons $\left[1200 + \int_0^{12} W(t) - R(t)\,dt \approx 1642.630\right]$
3. At $t = 6.495$ Because:
   at $t = 0$, there are 1200 gallons of water in the tank
   at $t = 6.495$, there are 525.242 gallons of water in the tank.
   at $t = 12$, there are 1642.630 gallons of water in the tank.
4. $\int_{12}^{k} R(t)\,dt = 1642.630$
5. $\dfrac{1}{12-0}\int_0^{12} (W(t) - R(t))dt = \dfrac{1642.630 - 1200}{12-0} = 36.917$ gallons/hour

ANALYZING A PARTICLE PROBLEM

PROBLEM #1 (NO CALCULATOR)

A particle moves along the x−axis with the velocity at time $t \geq 0$ given by $v(t) = -1 + e^{1-t}$.

1. Find the acceleration of the particle at time $t = 3$.

2. Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.

3. Find all values of *t* at which the particle changes direction. Justify your answer.

4. What is the average velocity of the particle over the interval $0 \leq t \leq 3$?

5. Find the total distance traveled by the particle over the interval $0 \leq t \leq 3$?

PROBLEM #2 (CALCULATOR)

A particle moves along the y−axis so that its velocity *v* at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$ and $y(0) = -1$

1. Find the acceleration of the particle at time $t = 2$.

2. Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

3. Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

4. Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

5. Find the total distance traveled by the particle over the interval $0 \leq t \leq 3$?

PROBLEM #1 ANSWERS:

1.	$a(t) = v'(t) = -e^{1-t} \quad \Rightarrow \quad a(3) = -e^{1-3} = -e^{-2}$
2.	$v(3) = -1 + e^{-2} < 0$ and $a(3) < 0 \quad \therefore$ Speed is increasing
3.	Particle changes direction at $t = 1$ because $v(t) = 0$ when $1 = e^{1-t}$, so $t = 1$. $v(t)$ changes from positive to negative at $t = 1$
4.	$\text{AV} = \frac{1}{3-0}\int_0^3 -1 + e^{1-t}\, dt \;=\; \frac{1}{3}\left[-t - e^{1-t}\right]_0^3 \;=\; \frac{1}{3}\left[\left(-3 - e^{-2}\right) - \left(0 - e\right)\right] \;=\; \frac{1}{3}\left(-3 - e^{-2} + e\right)$
5.	$\text{TD} = \int_0^1 -1 + e^{1-t} dt - \int_1^3 -1 + e^{1-t} dt = \left[-t - e^{1-t}\right]_0^1 - \left[-t - e^{1-t}\right]_1^3 =$ $\left[\left(-1 - e^0\right) - \left(0 - e\right)\right] - \left[\left(-3 - e^{-2}\right) - \left(-1 - e^0\right)\right] = \left[-2 + e - \left(-1 - e^{-2}\right)\right] = -1 + e + e^{-2}$

PROBLEM #2 ANSWERS:

1.	$a(2) = v'(2) = -0.133$
2.	$v(2) = -0.436$ Speed is increasing since $a(2)$ and $v(2)$ are both negative.
3.	$v(t) = 0$ when $t \approx 0.443$. $y(t)$ must have a maximum at $t \approx 0.443$ because $y'(t)$ changes from positive to negative at $t \approx 0.443$. Maximum doesn't occur at endpoints because $y(t)$ increases on the interval (0,0.443) and $y(t)$ decreases when $t \geq 0.443$.
4.	$y(2) = -1 + \int_0^2 v(t)\, dt = -1.361$. The particle is moving away from the origin since $v(2) < 0$ and $y(2) < 0$.
5.	$\text{TD} = \int_0^3 \lvert v(t) \rvert\, dt = 0.940$

EULER'S METHOD/APPROXIMATION

Question #1 (Calculator)

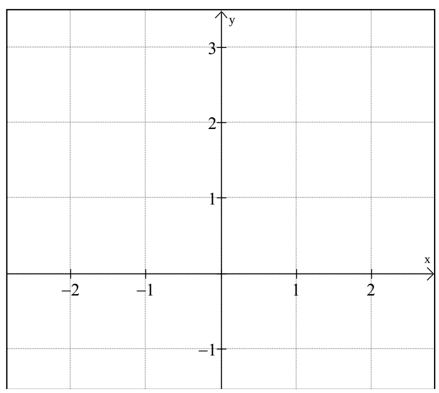
Let f be the function whose graph goes through the point $(3,6)$ and whose derivative is given by $f'(x) = \frac{1+e^x}{x^2}$

1. Write an equation of the line tangent to the graph of f at $x=3$ and use it to approximate $f(3.1)$.
2. Use Euler's method, starting at $x=3$ with a step size of 0.05, to approximate $f(3.1)$.
3. Is this approximation (from #2) less than or greater than $f(3.1)$? Why?
4. Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

Question #2 (No Calculator)

Consider the differential equation given by: $\frac{dy}{dx} = \frac{xy}{2}$

1. On the axes provided below, sketch a slope field for the given differential equation.



2. Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show work!
3. Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$

Question #3 (No Calculator)

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

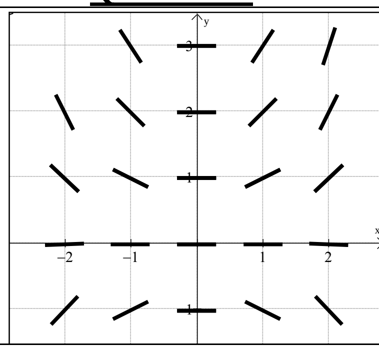
1. Evaluate $\int_1^{\infty} -3xf(x)dx$. Show the work that lead to your answer.
2. Use Euler's method, starting at $x=1$ with a step size of 0.5, to approximate $f(2)$.
3. Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

EULER'S METHOD/APPROXIMATION ANSWERS

Question #1

1	$f'(3)=\frac{1+e^3}{9}=\boxed{2.343} \Rightarrow \boxed{\boxed{y-6=\left(\frac{1+e^3}{9}\right)(x-3)}} \Rightarrow f(3.1)\approx 6+\left(\frac{1+e^3}{9}\right)(3.1-3)\approx \boxed{\boxed{6.234}}$
2	$f(3.05)\approx f(3)+f'(3)(0.05)=6+0.11714=\boxed{\boxed{6.11714}}$ $f(3.1)\approx 6.11714+f'(3.05)(.05)=6.11714+(2.37735)(0.05)=\boxed{\boxed{6.236}}$
3	$f''(x)=\frac{x^2e^x-2x\left(1+e^x\right)}{x^4}=\frac{(x-2)e^x-2}{x^3} \Rightarrow$ For $x\ge 3$, $f''(x)>0$ so the graph of f is concave upward on $(3,3.1)$. $\therefore$ the Euler approximation lines at 3 and 3.05 lie $\boxed{\boxed{\text{below}}}$ the graph. $\therefore$ $\boxed{\boxed{\text{Under approximates}}}$.
4	$\int_3^{3.1} f'(x)\,dx=f(3.1)-f(3) \Rightarrow f(3.1)=f(3)+\int_3^{3.1}f'(x)\,dx \Rightarrow f(3.1)=6+0.2378=\boxed{\boxed{6.238}}$

Question #2

1	
2	$f(0.1)\approx f(0)+f'(0)(0.1)=3+\left[\frac{(0)(3)}{2}\right](0.1)=\boxed{\boxed{3}}$ $f(0.2)\approx f(0.1)+f'(0.1)(0.1) \Rightarrow 3+\left[\frac{(0.1)(3)}{2}\right](0.1)=\boxed{\boxed{3.015}}$
3	$\frac{dy}{dx}=\frac{xy}{2} \Rightarrow \int\frac{dy}{y}=\int\frac{x}{2}dx \Rightarrow \ln\lvert y\rvert=\frac{1}{4}x^2+C \Rightarrow y=Ce^{x^2/4} \Rightarrow 3=Ce^0 \Rightarrow C=3 \Rightarrow \boxed{\boxed{y=3e^{x^2/4}}}$ $\therefore\ f(0.2)=3e^{\frac{(0.2)^2}{4}}=3e^{\frac{0.04}{4}}=3e^{0.01}\approx\boxed{\boxed{3.030}}$

Question #3

1	$\int_1^\infty -3x\,f(x)dx \Rightarrow \int_1^\infty f'(x)dx=\lim_{b\to\infty}\int_1^b f'(x)dx=\lim_{b\to\infty} f(x)\Big]_1^b=\lim_{b\to\infty}\left[f(b)-f(1)\right]=0-4=\boxed{\boxed{-4}}$
2	$f(1.5)\approx f(1)+\left[f'(1)\right](0.5) \Rightarrow 4+\left[(-3)(1)(4)\right](0.5)=\boxed{\boxed{-2}}$ $f(2)\approx -2+\left[f'(1.5)\right](0.5) \Rightarrow -2+\left[(-3)(1.5)(-2)\right](0.5)=\boxed{\boxed{2.5}}$
3	$\int\frac{dy}{y}=\int -3x\,dx \Rightarrow \ln y=-\frac{3}{2}x^2+k \Rightarrow y=Ce^{-3/2x^2} \Rightarrow 4=Ce^{-3/2} \Rightarrow C=4e^{3/2} \Rightarrow \boxed{\boxed{y=4e^{3/2}e^{-3/2x^2}}}$

IMPLICIT DIFFERENTIATION

PROBLEM #1 (NO CALCULATOR)

Consider the curve $x^2y - x^3y = 1$

1. Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{y(3x-2)}{x(1-x)}$

2. Find the equation of all horizontal tangent lines.

3. Find the equation of all vertical tangent lines.

4. Find the equation of the tangent line(s) at $x = 2$.

5. Using the tangent line at $x = 2$, approximate $y(2.1)$.

6. Is the curve increasing or decreasing at $x = \dfrac{1}{2}$? Justify your answer.

7. Is the curve concave up or down at $x = \dfrac{1}{2}$? Justify your answer.

8. Would a tangent line approximation <u>overestimate</u> or <u>underestimate</u> at $x = \dfrac{1}{2}$? Why?

PROBLEM #2 (NO CALCULATOR)

Consider the curve $xy^2 - x^2y = 2$

1. Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{y(2x-y)}{x(2y-x)}$

2. Find the equation of all horizontal tangent lines.

3. Find the equation of all vertical tangent lines.

4. Is the curve increasing or decreasing at $(1,-1)$? Justify your answer.

5. Is the curve concave up or down at $(1,-1)$? Justify your answer.

PROBLEM #1 ANSWERS:

2. $y=\frac{27}{4}$ {This occurs when $x=\frac{2}{3}$ }	5. $\frac{-1}{5}$
3. $x=0$ or $x=1$	6. Decreasing because $\frac{dy}{dx}=-16$ at $x=\frac{1}{2}$
4. $y+\frac{1}{4}=\frac{1}{2}(x-2)$	7. Concave up because $\frac{d^2y}{dx^2}=128$ at $x=\frac{1}{2}$
8. Underestimate because curve is concave up at $x=\frac{1}{2}$	

PROBLEM #2 ANSWERS:

1.
$$y^2+x(2y)\left(\frac{dy}{dx}\right)-[2xy+]=0 \;\Rightarrow\; y^2+(2xy)\left(\frac{dy}{dx}\right)-2xy-x^2\left(\frac{dy}{dx}\right)=0$$
$$\frac{dy}{dx}\left[2xy-x^2\right]=2xy-y^2 \;\Rightarrow\; \frac{dy}{dx}=\frac{2xy-y^2}{2xy-x^2}=\frac{y(2x-y)}{x(2y-x)}$$

2.
$$\frac{dy}{dx}=\frac{y(2x-y)}{x(2y-x)}=\frac{0}{1} \;\Rightarrow\; y(2x-y)=0 \;\Rightarrow\; y=0 \text{ or } 2x-y=0 \Rightarrow y=0 \text{ or } y=2x$$
$$\text{If } y=0, \text{ then } xy^2-x^2y=2 \;\Rightarrow\; x(0)^2-x^2(0)=2 \;\Rightarrow\; 0=2 \quad \text{BAD !!!!!}$$
$$\text{If } y=2x, \text{ then } x(2x)^2-x^2(2x)=2 \Rightarrow 4x^3-2x^3=2 \Rightarrow 2x^3=2 \;\Rightarrow\; x^3=1 \;\Rightarrow\; x=1$$
$$\text{If } x=1, \text{ then } xy^2-x^2y=2 \;\Rightarrow\; y^2-y=2 \Rightarrow y^2-y-2=0 \Rightarrow (y-2)(y+1)=0 \Rightarrow y=2 \text{ or } y=-1$$

3.
$$\boxed{\frac{dy}{dx}=\frac{y(2x-y)}{x(2y-x)}=\frac{1}{0} \;\Rightarrow\; x(2y-x)=0 \;\Rightarrow\; x=0 \text{ or } 2y-x=0 \Rightarrow x=0 \text{ or } x=2y}$$
$$\boxed{\begin{array}{l}\text{If } x=0, \text{ then } xy^2-x^2y=2 \;\Rightarrow\; 0y^2-0y=2 \;\Rightarrow\; 0=2 \quad \text{BAD !!!!!} \\ \text{If } x=2y, \text{ then } (2y)y^2-(2y)^2y=2 \Rightarrow 2y^3-4y^3=2 \Rightarrow -2y^3=2 \;\Rightarrow\; y^3=-1 \;\Rightarrow\; y=-1\end{array}}$$
$$\boxed{\text{If } y=-1, \text{ then } xy^2-x^2y=2 \Rightarrow x(-1)^2-x^2(-1)=2 \Rightarrow x^2+x-2=0 \Rightarrow (x+2)(x-1)=0 \Rightarrow x=-2 \text{ or } x=1}$$

4. Increasing because $\frac{dy}{dx} = \frac{y(2x-y)}{x(2y-x)} = \frac{-1(2+1)}{1(-2-1)} = \frac{-3}{-3} = 1 > 0 \quad \left[\frac{dy}{dx}=1\right]$

5.
$$\frac{dy}{dx}=\frac{2xy-y^2}{2xy-x^2} \;\Rightarrow\; \frac{dy}{dx}(1,-1)=\frac{2(1)(-1)-(-1)^2}{2(1)(-1)-(1)^2}=\frac{-2-1}{-2-1}=\frac{-3}{-3}=1$$
$$\square = 2xy-y^2 \;\Rightarrow\; \square(1,-1) = 2(1)(-1)-(-1)^2 = -2-1 = -3$$
$$\Delta = 2xy-x^2 \Rightarrow \Delta(1,-1) = 2(1)(-1)-(1)^2 = -2-1 = -3$$
$$\square' = 2y+2x\left(\frac{dy}{dx}\right)-2y\left(\frac{dy}{dx}\right) \;\Rightarrow \square'(1,-1) = 2(-1)+2(1)(1)-2(-1)(1) = -2+2+2=2$$
$$\Delta' = 2y+2x\left(\frac{dy}{dx}\right)-2x \;\Rightarrow \Delta'(1,-1) = 2(-1)+2(1)(1)-2(1) = -2+2-2 = -2$$
$$\therefore\; \frac{d^2y}{dx^2} = \frac{\square'\Delta-\square\Delta'}{\Delta^2}=\frac{(2)(-3)-(-3)(-2)}{(-3)^2}=\frac{-12}{9}=\frac{-4}{3} \;\therefore\; \text{Concave Down} \left[\frac{d^2y}{dx^2}<0\right]$$

LOGISTIC GROWTH

Question #1

A certain rumor spreads through a community at the rate of $\frac{dy}{dt} = 2y(1-y)$ where y is the proportion of the population that has heard the rumor at time t.

1. What proportion of the population has heard the rumor when it is spreading the fastest?

2. If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

3. At what time t is the rumor spreading the fastest?

LOGISTIC GROWTH
(ANSWERS)

1	y is growing the fastest when y = one-half of the carrying capacity, A. So $y = \frac{1}{2}\text{A} = \frac{1}{2}(1) = 1 \Rightarrow$ proportion when rumor is spreading the fastest is $\boxed{y = \frac{1}{2}}$
2	Logistic growth differential equation is $\frac{dy}{dt} = ky(A - y)$ and in this problem $\frac{dy}{dt} = 2y(1 - y)$ So $A = 1$ and $k = 2$. The solution is $y = \frac{1}{1 + Ce^{-Akt}}$ So in this problem $y = \frac{1}{1 + Ce^{-2t}}$ Since $y = 0.1$ when $t = 0 \Rightarrow 0.1 = \frac{1}{1 + Ce^{0}} = \frac{1}{1 + C} \Rightarrow 0.1 + 0.1C = 1 \Rightarrow C = 9$ So $\boxed{y = \frac{1}{1 + 9e^{-2t}}}$
3	From #1, the rumor spreads the fastest when $y = \frac{1}{2}$. $\frac{1}{2} = \frac{1}{1 + 9e^{-2t}} \Rightarrow 1 + 9e^{-2t} = 2 \Rightarrow e^{-2t} = \frac{1}{9} \Rightarrow -2t = \ln 9 \Rightarrow \boxed{t = \frac{\ln 9}{-2}}$ Leave the answer in this form if it is a free response problem, but it does simplify to $\ln 3$.

GRAPHING

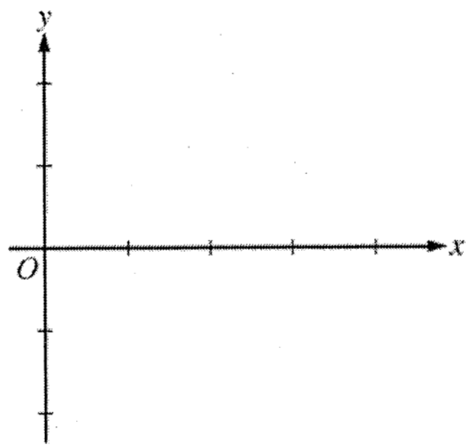
PROBLEM #1 (NO CALCULATOR)

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.



(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.

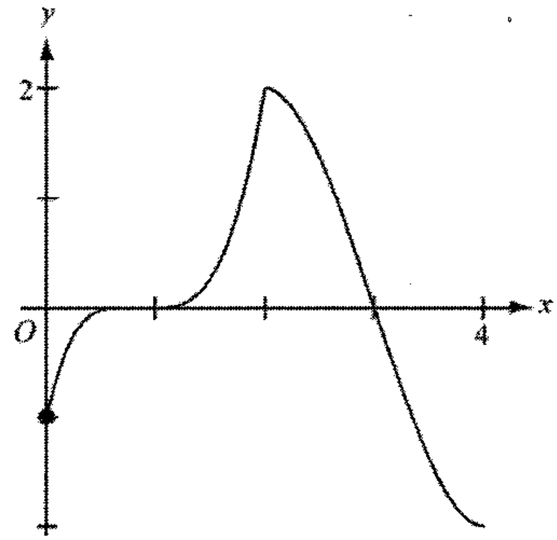
(d) For the function g defined in part (c), find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.

(e) Set up, but do not evaluate, an expression that would result in the area of f (x) for $0 < x < 4$.

PROBLEM #1 ANSWERS:

(a) f has a relative maximum at $x = 2$ because f' changes from positive to negative at $x = 2$.

(b)



(c) $g'(x) = f(x) = 0$ at $x = 1, 3$.
g' changes from negative to positive at $x = 1$ so g has a relative minimum at $x = 1$. g' changes from positive to negative at $x = 3$ so g has a relative maximum at $x = 3$.

(d) The graph of g has a point of inflection at $x = 2$ because $g'' = f'$ changes sign at $x = 2$.

(e) Area $= -\int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx - \int_3^4 f(x)\,dx$