Practice Problems for Midterm 2

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(1) Consider the integral
$$\int_0^{\pi} \sec^2(x)dx.$$

(a) Explain where the following logic goes wrong:
$$\begin{aligned}\int_0^{\pi} \sec^2(x)dx &= \tan(x)\Big|_0^{\pi} \\ &= \tan(\pi) - \tan(0) \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

(b) Evaluate $\int_0^{\pi} \sec^2(x)dx$.

(2) (a) Assuming $f'(x)$ is continuous, evaluate
$$\int_{-5}^{5} xf'(x^2)dx.$$

(b) Now assuming that $f(4) = 2$ and $f(0) = 0$, evaluate
$$\int_0^2 xf'(x^2)dx.$$

(c) If $f'(1) = 0$, $f(1) = 2$ and $f(0) = -1$, evaluate
$$\int_0^1 xf''(x)dx.$$

(3) Suppose $f(x)$ is continuous on $[a, \infty)$. Define what it means for the integral
$$\int_a^{\infty} f(x)dx$$
to converge. Define what it means for the integral above to diverge.

(4) For each of the following, explain why the integral is improper, and then determine if the integral converges or diverges by evaluating the integral.

(a) $\int_0^1 \ln(x)dx$

(b) $\int_{\pi}^{\infty} \sin(2x+3)dx$

(c) $\int_0^{\infty} \frac{1}{x^2-1}dx$

(d) $\int_0^2 \frac{x}{\sqrt{4-x^2}}dx$

(e) $\int_0^\infty \tan^{-1}(x)dx$

(f) $\int_{-\infty}^\infty \frac{1}{4+x^2}dx$

(g) $\int_{-3}^{-1} \frac{1}{x^2+2x}dx$.

(h) $\int_1^2 \frac{x}{\sqrt[3]{x^2-4}}dx$

(i) $\int_{-\infty}^0 \frac{e^{1/x}}{x^2}dx$

(5) Explain why the integrals

$$\int_0^\infty \sin^{2013}(x)dx \quad \text{and} \quad \int_1^\infty \sqrt{x^2-1}dx$$

both diverge. (Yes, you can do this without integrating them.)

(6) Give an example of a continuous odd function $f(x)$ such that

$$\int_{-\infty}^\infty f(x)dx$$

is not zero.

(7) Give an example of how

$$\int_{-\infty}^\infty f(x)dx \neq \lim_{M\to\infty} \int_{-M}^M f(x)dx.$$

(That is, find a function $f(x)$ that illustrates this idea.)

(8) Evaluate the following integrals using any method. Show all work.

(a) $\int 5x(2+x^2)^3dx$

(b) $\int 7xe^{x^2}dx$

(c) $\int x(3-x)^{2013}dx$

(d) $\int_0^1 x^2e^{-4x}dx$

(e) $\int \frac{1}{x^2+2}dx$

(f) $\int \frac{1}{x^2+2x+5}dx$

(g) $\int \frac{x}{x^2+6x+13}dx$

(h) $\int \frac{x^3+1}{x^2(x-1)}dx$

(i) $\int \sin^{-1}(2x)dx$

(j) $\int_0^1 x\tan^{-1}(x)dx$

(k) $\int x\ln(xe^x)dx$

(l) $\int x^3e^{x^2}dx$

(m) $\int x^8\sqrt{1+x^3}dx$

(n) $\int \frac{dx}{e^x+1}$ (Hint: Do a u-sub with $u = e^x$.)

(o) $\int_0^{9\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$

(p) $\int_0^{1/2} \frac{dx}{1+4x^2}dx$

(q) $\int \sin(\sqrt{x})dx$

(r) $\int 2x\cot(x^2)dx$

(s) $\int e^{\sqrt[3]{x}}dx$

(t) $\int \frac{x+1}{x(x^2+1)}dx$

(u) $\int \frac{3x+5}{x^2+3x+2}dx$

(v) $\int \frac{x^3+1}{(x-1)^2(x+3)}dx$

(w) $\int \sqrt[4]{1+\sqrt{x}}dx$

(x) $\int_0^{\pi} \sin(2x)\cos(3x)dx$

(y) $\int e^x\sin(2x)dx$

(z) $\int \frac{1}{x\ln(x)\ln(\ln(x))}dx$

(A) $\int \tan^5(2x)\sec^2(2x)dx$

(B) $\int \sin^2(x)\cos^5(x)dx$

(C) $\int_{-\pi}^{\pi} \cos^{1000}(x)\sin^{2013}(x)dx$

(9) Jack and Jill went up the hill to fetch a pail of water, but instead they found themselves trying to integrate $f(x) = \sin(x)\cos(x)$, i.e. they wanted to compute

$$\int \sin(x)\cos(x)dx$$

(yes, this is how the nursery rhyme goes.)

(a) Jack used a u-sub with $u = \sin(x)$. What answer did he get for the integral?

(b) Jill did a u-sub with $u = \cos(x)$. What answer did she get for the integral?

(c) Then they found they got different answers, at which point Jack was so confused that he fell down and broke his crown. Jill almost went tumbling after, until she realized the answers were not so different. Explain why your answers for (a) and (b) are, in fact, the same.

(10) Determine whether the following integrals converge or diverge. Do NOT just guess. If the integral converges, you are NOT required to evaluate it. However, if you use the comparison test, you must show the integral you use for the comparison either converges or diverges.

(a) $\int_5^{\infty} \frac{\cos^2(x)}{x^2}dx$

(b) $\int_{1/2}^{\infty} \frac{1+\sin^4(x)}{\sqrt{x}} dx$

(c) $\int_1^{\infty} \frac{2}{\sqrt{4x^2-2}} dx$

(d) $\int_0^{\pi/2} \frac{2013}{x\sin(x)} dx$

(e) $\int_3^{\infty} \frac{1}{xe^x} dx$

(f) $\int_0^{\infty} \frac{1}{x^2+\sqrt{x}} dx$

(g) $\int_3^{\infty} \frac{1}{x-e^{-x}} dx$

(h) $\int_1^{\infty} e^{-x^2} dx$

(i) $\int_0^{\infty} e^{-x^2} dx$

(j) $\int_1^{\infty} \frac{x-1}{x^4+2x^2} dx$

(k) $\int_6^{\infty} \frac{x^2+1}{x^3(\cos^2(x)+1)} dx$

(l) $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}} dx$

(11) Solve the following differential equations. If initial conditions are provided, find the unique solution. Solve explicitly for y unless otherwise told.

(a) $\frac{dy}{dx} = xy$, $y(0) = 5$.

(b) $\frac{dy}{dx} = y^2 e^x$, $y(0) = 1$

(c) $\frac{dy}{dx} = \sec(y)\sin(x)$. Do not solve explicitly for y.

(d) $\frac{dy}{dx} = 5xe^{x^2+1}$

(e) $\frac{dy}{dx} = \frac{xy+y}{x^2+1}$, $y(0) = 1$

(f) $\frac{dy}{dx} = \frac{x(x^2+1)}{y^3}$, $y(0) = -1$

(g) $\frac{dy}{dx} = xy(y-2)$, $y(0) = 1$

(h) $\frac{dy}{dx} = xy(y-2)$, $y(0) = 2$

(i) $\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2)$. Do not solve explicitly for y.

(j) $(1+e^x)\frac{dy}{dx} - \cot^2(y) = 0$. Do not solve explicitly for y.

(k) $\frac{dy}{dx} = y^2 x - 1 - y^2 + x$

(l) $\frac{dy}{dx} = \frac{\ln(xy^2)}{\ln(y)} - 2$. Do not solve explicitly for y.

(m) $\frac{dy}{dx} = e^{-\sqrt{y}}$, $y \geq 0$, with $y(0) = 4$. Do not solve explicitly for y.

(12) Consider the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{100}\right).$$

Solve the differential equation explicitly for P with initial conditions $P(0) = 150$, $P(0) = 50$, and $P(0) = 100$ (that is, find three specific solutions, one for each of the initial conditions).

(13) Consider the differential equation

$$\frac{dP}{dt} = f(P) = 16P - P^3.$$

Solve the differential equation, and find specific solutions in the cases: $P(0) = 2$, $P(0) = 5$, $P(0) = -1$, and $P(0) = 0$ (you do not need to find explicit solutions).

(14) Here is a little partial fractions practice. Integrate each of the following rational functions. Formulas you may or may not use here (but which you should know for the exam):

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

(a) $f(x) = \frac{x+1}{x(x-1)(2x-1)}$

(b) $f(x) = \frac{x^2+3x+10}{x^4-1}$

(c) $f(x) = \frac{1}{(x^2-3x+2)^2}$

(d) $f(x) = \frac{x^3+x^2+4x+3}{(x^2+3)^2}$

(e) $f(x) = \frac{4x^2+7x+2}{x^3+2x^2+2x}$

(f) $f(x) = \frac{x^2-x+1}{(x+1)^3}$

Answers/Hints:

(1) (a) The function $f(x) = \sec^2(x)$ is not continuous on $[0, \pi]$, so we cannot apply the Fundamental Theorem of Calculus.

(b) The integral diverges.

(2) (a) 0 (the function is odd and the interval $[-5, 5]$ is symmetric about 0).

(b) 1 (Do a u-sub with $u = x^2$, so that the integral becomes $\frac{1}{2}\int_0^4 f'(u)du$).

(c) -3 (Do integration by parts with $u = x$ and $dv = f''(x)dx$).

(3) The integral converges if the limit

$$\lim_{M\to\infty}\int_a^M f(x)dx$$

exists and is finite. The integral diverges if the limit is infinite.

(4) For each of the following, explain why the integral is improper, and then determine if the integral converges or diverges by evaluating the integral.

(a) -1

(b) Diverges

(c) Diverges

(d) 2

(e) Diverges

(f) $\pi/2$

(g) Diverges

(h) $-\frac{3}{4}\sqrt[3]{9}$

(i) 1

(5) Both functions $\sin^{2013}(x)$ and $\sqrt{x^2 - 1}$ do not approach 0 as $x \to \infty$.

(6) $f(x) = x$ works since the integral will diverge in this case.

(7) $f(x) = x$, since the right hand side will be 0 but the integral on the left hand side does not exist.

(8) (a) $\frac{5}{8}(x^2 + 2)^4 + C$

(b) $\frac{7}{2}e^{x^2} + C$

(c) $-\frac{3}{2014}(3 - x)^{2014} + \frac{1}{2015}(3 - x)^{2015} + C$

(d) $\frac{1-13e^{-4}}{32}$

(e) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

(f) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$

(g) $\frac{1}{2}\ln|x^2+6x+13|-\frac{3}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$

(h) $x+\frac{1}{x}+2\ln|x-1|-\ln|x|+C$

(i) $x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+C$

(j) $\frac{\pi}{4}-\frac{1}{2}$

(k) $\frac{x^3}{3}-\frac{x^2}{2}\ln|x|-\frac{x^2}{4}+C$

(l) $\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$

(m) $\frac{2}{21}(x^3+1)^{7/2}-\frac{4}{15}(x^3+1)^{5/2}+\frac{2}{9}(x^3+1)^{3/2}+C$

(n) $x-\ln|e^x+1|+C$

(o) 4

(p) $\pi/8$

(q) $2\sin(\sqrt{x})-2\sqrt{x}\cos(\sqrt{x})+C$

(r) $\ln|\sin(x^2)|+C$

(s) $3x^{2/3}e^{\sqrt[3]{x}}-6\sqrt[3]{x}e^{\sqrt[3]{x}}+6e^{\sqrt[3]{x}}+C$

(t) $-\frac{1}{2}\ln(x^2+1)+\ln|x|+\tan^{-1}(x)+C$

(u) $\ln|x+2|+2\ln|x+1|+C$

(v) $x+\frac{5}{8}\ln|x-1|-\frac{1}{2(x-1)}-\frac{13}{8}\ln|x+3|+C$

(w) $\frac{8}{9}(1+\sqrt{x})^{9/4}-\frac{8}{5}(1+\sqrt{x})^{5/4}+C$

(x) $-4/5$

(y) $\frac{e^x\sin(2x)-2e^x\cos(2x)}{5}+C$

(z) $\ln|\ln(\ln(x))|+C$

(A) $\frac{1}{12}\tan^6(2x)+C$

(B) $\frac{\sin^3(x)}{3}-\frac{2}{5}\sin^5(x)+\frac{1}{7}\sin^7(x)+C$

(C) 0

(9) (a) $\frac{\sin^2(x)}{2}+C$

(b) $-\frac{\cos^2(x)}{2}+C$

(c) Recall that $\sin^2(x)=1-\cos^2(x)$, so Jack's answer in (a) is the same thing as

$$\frac{\sin^2(x)}{2}+C=\frac{1-\cos^2(x)}{2}+C=\frac{1}{2}-\frac{\cos^2(x)}{2}+C=-\frac{\cos^2(x)}{2}+C,$$

which is the answer in (b).

(10) (a) Converges (compare with $1/x^2$)

(b) Diverges (compare with $1/\sqrt{x}$)

(c) Diverges (compare with $2/x$)

(d) Diverges (compare with $2013/x$)

(e) Converges (compare with $1/e^x$)

(f) Converge (compare with $1/\sqrt{x}$ on $(0,1]$ and $1/x^2$ on $[1,\infty)$)

(g) Diverge ($1/x$)

(h) Converge (e^{-x})

(i) Converge (e^{-x} on $[1,\infty)$)

(j) Converge ($1/x^3$)

(k) Diverge ($\frac{1}{2x}$)

(l) Converge (e^x on $(-\infty,0)$ and e^{-x} on $(0,\infty)$)

(11) (a) $y = 5e^{x^2/2}$

(b) $y = -\frac{1}{e^x-2}$

(c) $\sin(y) = -\cos(x) + C$

(d) $y = \frac{5}{2}e^{x^2+1} + C$

(e) $y = e^{\tan^{-1}(x)}\sqrt{x^2+1}$

(f) $y = -\sqrt[4]{x^4+2x^2+1}$

(g) $y = \frac{2}{e^{x^2}+1}$

(h) $y = 2$

(i) $\frac{e^{-y}(\sin(y)-\cos(y))}{2} = -e^{-t}(t^2+2t+3) + C$

(j) $\tan(y) - y = x - \ln(e^x+1) + C$

(k) $y = \tan\left(\frac{x^2}{2} - x + C\right)$

(l) $y\ln(y) - y = x\ln(x) - x + C$

(m) $2(\sqrt{y}e^{\sqrt{y}} - e^{\sqrt{y}}) = t + 2e^2$.

(12) We get

$$P(t) = \frac{Ce^t}{1 + \frac{1}{100}Ce^t},$$

or any simplified version of this. When $P(0) = 150$, $C = -300$. If $P(0) = 50$, then $C = 100$. So in each of these cases, we plug these values for C into the equation above to get $P(t)$. In the case $P(0) = 100$, we run into trouble, because we can't solve for C. However, we note that we divided by 0 in the beginning, and so the unique solution is the constant solution $P(t) = 100$ when $P(0) = 100$.

(13) The general solution is

$$\frac{1}{16}\ln|y| - \frac{1}{32}\ln|4+y| - \frac{1}{32}\ln|4-y| = t + C.$$

In the case $P(0) = 2$, $C = \frac{1}{32}\ln(1/3)$. When $P(0) = 5$, $C = \frac{1}{16}\ln(5) - \frac{1}{32}\ln(9)$. When $P(0) = -1$, $C = \frac{1}{32}\ln(3/5)$, and when $P(0) = 0$, the only solution is $P(t) = 0$.

(14) (a) $\ln|x| - 3\ln|2x-1| + 2\ln|x-1| + C$

(b) $-\tan^{-1}(x) - \ln|x+1| + \ln|x-1| + C$

(c) $2\ln|x-1| - \frac{1}{x-1} - 2\ln|x-2| - \frac{1}{x-2} + C$

(d) $-\frac{1}{2(x^2+3)} + \frac{1}{2}\ln|x^2+3| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

(e) $\ln|x| + \frac{3}{2}\ln|x^2+2x+2| + 2\tan^{-1}(x+1) + C$

(f) $\ln|x+1| + \frac{3}{x+1} - \frac{3}{2(x+1)^2} + C$