Proof of Heron's Formula: (Definitely NOT the way Heron did it!)
Recall the law of cosines:

$c^2 = a^2 + b^2 - 2ab\cos C$ so that $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Recall the two half angle formulas:

$\cos\dfrac{C}{2} = \sqrt{\dfrac{1+\cos C}{2}}$ and $\sin\dfrac{C}{2} = \sqrt{\dfrac{1-\cos C}{2}}$

Consider $\cos\dfrac{C}{2} = \sqrt{\dfrac{1+\cos C}{2}} = \sqrt{\dfrac{1+\left(\dfrac{a^2+b^2-c^2}{2ab}\right)}{2}} = \sqrt{\dfrac{a^2+2ab+b^2-c^2}{4ab}} = \sqrt{\dfrac{(a+b)^2-c^2}{4ab}} =$

$$\sqrt{\frac{(a+b-c)(a+b+c)}{4ab}} \qquad (1)$$

Now we need to introduce the semiperimeter $s = \frac{1}{2}(a+b+c)$. Since $(a+b-c) = (a+b+c) - 2c = 2s - 2c$, and $(a+b+c) = 2s$, we can substitute into our previous result (1):

$\sqrt{\dfrac{(a+b-c)(a+b+c)}{4ab}} = \sqrt{\dfrac{(2s-2c)2s}{4ab}} = \sqrt{\dfrac{4s(s-c)}{4ab}} =$

$$\sqrt{\frac{s(s-c)}{ab}} \qquad (2)$$

Similarly, we can now consider

$\sin\dfrac{C}{2} = \sqrt{\dfrac{1-\cos C}{2}} = \sqrt{\dfrac{1-\left(\dfrac{a^2+b^2-c^2}{2ab}\right)}{2}} = \sqrt{\dfrac{-a^2+2ab-b^2+c^2}{4ab}} = \sqrt{\dfrac{c^2-(a-b)^2}{4ab}} =$

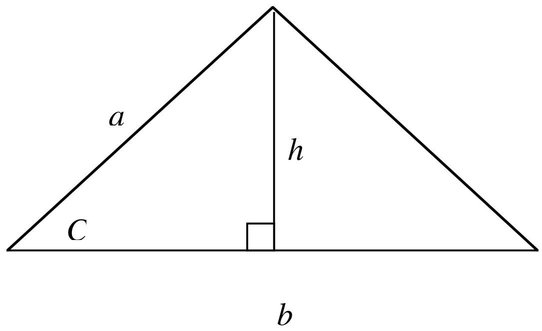
$\sqrt{\dfrac{(c-(a-b))(c+(a-b)}{4ab}} = \sqrt{\dfrac{(c-a+b)(c+a-b)}{4ab}}$

Introducing s into the picture, we can see $(c-a+b) = (a+b+c) - 2a = 2s - 2a$ and $(c+a-b) = (a+b+c) - 2b = 2s - 2b$, so substituting this into our previous result,

$\sqrt{\dfrac{(c-a+b)(c+a-b)}{4ab}} = \sqrt{\dfrac{(2s-2a)(2s-2b)}{4ab}} = \sqrt{\dfrac{2(s-a)2(s-b)}{4ab}} = \sqrt{\dfrac{4(s-a)(s-b)}{4ab}} =$

$$\sqrt{\frac{(s-a)(s-b)}{ab}} \qquad (3)$$

The area of a triangle is half the base times the altitude. $\frac{1}{2}bh$



since $\sin C = \dfrac{h}{a}$ then $h = a\sin C$

So the area of the triangle :

$$K = \tfrac{1}{2}ab\sin C \qquad (4)$$

Now this can be rewritten as

$$\tfrac{1}{2}ab \cdot \sin 2\left(\frac{C}{2}\right)$$

$$= \tfrac{1}{2}ab \cdot 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= ab\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)$$

Substituting from our result from (2) and (3),

$$= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= ab\sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2b^2}}$$

$$= ab\frac{\sqrt{s(s-a)(s-b)(s-c)}}{ab}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

as Heron told us when he lived in Alexandria, but he was so clever, he didn't need trigonometry to prove it. An optional project: Make a written report, web-page, video or PowerPoint presentation of Heron's Classic Proof, or a proof of the Law of Cosines or the Law of Sines.