Induction WS **Name:** Per:

Explanation

Mathematical Induction is a method of proof. We use this method to prove certain propositions involving positive integers. Mathematical Induction is based on a property of the natural numbers, $\mathbb{N}$, called the Well Ordering Principle which states that evey nonempty subset of positive integers has a least element.

There are three steps in the method:

1. Prove the statement is true at the starting point (usually $n = 1$).
2. Suppose (by hypothesis) that the statement is true for some integer k, and prove that if it is true for k, it must also be true for $k + 1$.
3. Since it is true for $n = 1$ (step 1) and it is true for the next step (step 2), then it is true for all $n = 1, 2, 3, \ldots$

Example

Prove $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$

1. We want to show this is true at the starting point $n = 1$.
   LHS $= 1$
   RHS $= 1^2 = 1$
   Since LHS=RHS, the statement is true for $n = 1$.
2. Suppose the statement is true for an integer k (that is, $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$)
   Now we want to show this is true for $k + 1$.
   In other words, we want to show
   $1+3+5+\cdots+(2k-1)+[2(k+1)-1] = (k+1)^2$.

   $$\begin{aligned} LHS &= 1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= k^2 + [2(k + 1) - 1] \text{ (by hypothesis)} \\ &= k^2 + (2k + 2 - 1) \\ &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \\ &= RHS \end{aligned}$$

   So, if the statement is true for k, it is true for $k + 1$.
3. The above together imply that the statement is true for all $n \in \mathbb{N}$ (all natural numbers) by induction.

Now You Try!

1. Prove that $2 + 4 + 6 + \cdots + 2n = n^2 + n$
2. Prove that $1+4+7+10+\cdots+3n-2 = \frac{n}{2}(3n-1)$
3. Prove that $1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n + 1)$

4. Prove that $2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$

5. Prove that $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$

A sneaky trick to help you do the second step of induction....

Very often it helps to do some work on some scratch paper. For example, lets consider proving

$$3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$$

.
After doing the first step, I write on some scratch paper the "goal", or what the RHS looks like if I substitute $(k+1)$ every place there is an k on the RHS. In this case,

$$(k+1)[(k+1)+2]$$

Next I combine terms a bit...

$$
\begin{aligned}
&(k+1)[(k+1)+2] && (1)\\
&= (k+1)(k+3) && (2)\\
&= k^2 + 4k + 3 && (3)
\end{aligned}
$$

Now that looks like something normal. Now I start the second step on my actual proof (putting aside my scratch paper), with the LHS:

$$
\begin{aligned}
&3 + 5 + 7 + \cdots + [2k+1] + [2(k+1)+1]\\
&= k(k+2) + [2(k+1)+1]\\
&= k^2 + 2k + 2k + 2 + 1\\
&= k^2 + 4k + 3
\end{aligned}
$$

Now doesn't that last line look familiar? Wasn't that line (3) from the scratch paper? So now we can finish it off (going in reverse from the scratch paper) with lines (2) and (1), and Step 2 is done!

$$
\begin{aligned}
&3 + 5 + 7 + \cdots + [2k+1] + [2(k+1)+1]\\
&= k(k+2) + [2(k+1)+1]\\
&= k^2 + 2k + 2k + 2 + 1\\
&= k^2 + 4k + 3\\
&= (k+1)(k+3)\\
&= (k+1)[(k+1)+2]
\end{aligned}
$$

Therefore if true for k it will be true for $k+1$, and if it is true for $n = 1$, if $k = n = 1$, we know it works for $n = 2$ and in fact all the rest of the natural numbers $n = 1, 2, 3, 4, \ldots$