BC Calc-Lagrange Remainder

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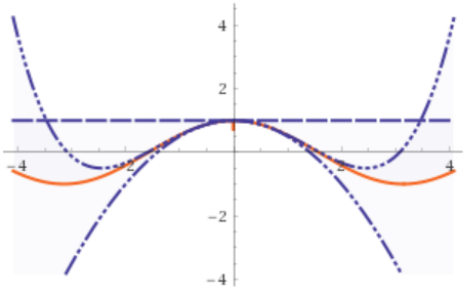


(order n approximation shown with n dots)

Let us define $P_N(x)$ the N^{th} degree Taylor polynomial around c :

$$P_N(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \ldots$$

If we define the Remainder function $R_N(x)$:

$$R_N(x) = f(x) - P_N(x)$$

then a version of the Taylor Remainder Theorem is the Lagrange Remainder:

$$|R_N(x)| \leq \left| \frac{M(x-c)^{N+1}}{(N+1)!} \right|$$

Proof

Before the proof, let us consider the following lemma:

$$\left| \int f(x)dx \right| \leq \int |f(x)|\, dx$$

To convince yourself that this is true, consider how if f is both positive and negative, the right term areas will accumulate and the left term areas cancel.

Now let us consider $R_N(x) = f(x) - P_N(x)$. Because it is a Taylor series around c we know

$$\begin{aligned}
R_N(c) &= f(c) - P_N(c) = 0 \\
R_N'(c) &= f'(c) - P_N'(c) = 0 \\
R_N''(c) &= f''(c) - P_N'''(c) = 0 \\
R_N'''(c) &= f'''(c) - P_N'''(c) = 0 \\
&\ldots \\
R_N^{(n)}(c) &= f^{(n)}(c) - P_N^{(n)}(c) = 0 \\
&\ldots \\
R_N^{(N)}(c) &= f^{(N)}(c) - P_N^{(N)}(c) = 0
\end{aligned}$$

Going beyond the N^{th} derivative we have

$$R_N^{(N+1)}(c) = f^{(N+1)}(c) - P_N^{(N+1)}(c)$$

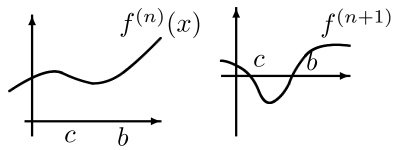
but because we are taking a derivative beyond the degree of the Taylor polynomial, $P_N^{(N+1)}(c) = 0$, we have

$$R_N^{(N+1)}(c) = f^{(N+1)}(c)$$

We want to show that we can bound this by some number M such that

$$\left|R_N^{(N+1)}(c)\right| = \left|f^{(N+1)}(c)\right| \leq M$$

Let consider $f^{(n+1)}(x)$ on the interval $[c, b]$. If $f^{(n+1)}(c)$ or $f^{(n+1)}(b)$ doesn't already equal M, it there will be a maximum value M by Rolle's theorem if continuous on $[c, b]$ and differentiable on (c, b).



Since

$$\left|f^{(n+1)}(x)\right| \leq M$$

where $x \in [c, b]$, we have

$$R_N^{(n+1)}(x) \leq M$$

Next we continue to integrate until we get to $|R_N(x)|$:

$$\int \left|R_N^{(n+1)}(x)\right| dx \leq \int M dx$$

by the lemma

$$\left|\int R_N^{(n+1)}(x) dx\right| \leq \int \left|R_N^{(n+1)}(x)\right| dx \leq \int M dx$$

so we can write

$$\left|\int R_N^{(n+1)}(x) dx\right| \leq \int M dx$$

$$\left|R_N^{(n)}(x)\right| \leq Mx + K$$

where K is a constant, which we wish to minimize. Recall $\left|R_N^{(c)}\right|$ can be 0. so $0 < Mc + K$ suggests $-Mc < K$ so that

$$\left|R_N^{(n)}(x)\right| \leq Mx - Mc = M(x - c)$$

As we continue to integrate

$$\int \left|R_N^{(n)}(x)\right| dx \leq \int M(x - c) dx$$

$$\left|R_N^{(n-1)}(x)\right| \leq \frac{M(x - c)^2}{2}$$

where the constant K can be minimized to 0 since $0 \leq K$. Next we integrate again.

$$\int \left|R_N^{(n-1)}(x)\right| dx \leq \int \frac{M(x - c)^2}{2} dx$$

$$\left|R_N^{(n-2)}(x)\right| \leq \frac{M(x - c)^3}{3 \cdot 2}$$

and again

$$\int \left|R_N^{(n-2)}(x)\right| dx \leq \int \frac{M(x - c)^3}{3 \cdot 2} dx$$

$$\left|R_N^{(n-3)}(x)\right| \leq \frac{M(x - c)^4}{4 \cdot 3 \cdot 2}$$

arriving eventually at the remainder function $R_N(x)$ when for some w, $n - w = 0$ (so $n = N$), we get to :

$$|R_N(x)| \leq \frac{M(x - c)^{N+1}}{(N + 1)!}$$

Use the Remainder Theorem to bound the error involved in using the specific Taylor polynomial, centered at 0, to approximate $f(x)$ at the given value.

1. $P_5(x)$ for $f(x) = \cos x$ at $x = 0.2$

2. $P_4(x)$ for $f(x) = e^x$ at $x = 0.8$

Use the Remainder Theorem to bound the error involved in using the specific Taylor polynomial, at the given center, to approximate $f(x)$ at the given value.

3. $P_2(x)$ for $f(x) = x^{5/2}$ centered at 1. Approximate $f(1.7)$

4. $P_3(x)$ for $f(x) = \dfrac{1}{1-x}$ centered at 2. Approximate $f(2.4)$

Determine the degree of the Taylor polynomial, centered at 0, that would be required to approximate the function at the given point to within the stated accuracy.

5. $f(x) = x\ln(1 + x)$, at $x = -0.2$, within $1/1000$.

6. $f(x) = \sin x$, at $x = 1$, within 1/1000.

7. $f(x) = e^{2x}$, at $x = 0.5$, within 1/100.

Answers

1.

2. Since $0 < z < 1$, max $f^{(5)}(z) = e$
$|R_4(0.8)| \leq \left|\dfrac{(0.8)^5}{5!}e\right| = .00742272$ so $e^{(0.8)}$ is about $2.2224 \pm .00742272$

3. Since $1 < z < 1.7$, max $f^{(3)}(z) = \dfrac{15}{8}$
$|R_2(1.7)| \leq 0.1071875$ or $3.66875 < f(1.7) < 3.7759375$

4. Since $2 < z < 2.4$, max $f^{(4)}(z) = 24$
$|R_3(2.4)| \leq 0.0256$ or $-0.7216 < f(2.4) < -0.696$

5. $f'(x) = \frac{x}{x+1} + \ln(x+1)$
$f''(x) = \frac{1}{(x+1)^2} + \frac{1}{x+1} = \frac{x+2}{(x+1)^2}$
$f'''(x) = -\frac{2}{(x+1)^3} - \frac{1}{(x+1)^2} = (-1)\left(\frac{x+3}{(x+1)^3}\right)$
$f^{(4)}(x) = \frac{6}{(x+1)^4} + \frac{2}{(x+1)^3} = \frac{3!}{(x+1)^4} + \frac{2!}{(x+1)^3}$
$f^{(5)}(x) = -\frac{24}{(x+1)^5} - \frac{6}{(x+1)^4} = (-1)\left(\frac{4!}{(x+1)^5} + \frac{3!}{(x+1)^4}\right)$
$f^{(n)}(x) = (-1)^n\left(\frac{(n-1)!}{(x+1)^n} + \frac{(n-2)!}{(x+1)^{n-1}}\right)$
$f^{(n+1)}(x) = (-1)^{n+1}\left(\frac{n!}{(x+1)^{n+1}} + \frac{(n-1)!}{(x+1)^n}\right)$
So we need to solve for n so that $f^{(n+1)}$ **term** is less than $\frac{1}{1000}$:
$\left|\dfrac{(-0.2)^{n+1}}{(n+1)!} \cdot \left(\dfrac{n!}{(z+1)^{n+1}} + \dfrac{(n-1)!}{(z+1)^n}\right)\right| \leq \dfrac{1}{1000}$
$\left|\dfrac{(-0.2)^{n+1}}{(n+1)!} \cdot (n! + (n-1)!)\right| \leq \dfrac{1}{1000}$
$\dfrac{(-0.2)^{n+1}}{n+1} + \dfrac{(-0.2)^{n+1}}{(n+1)n} \leq \frac{1}{1000}$
$\dfrac{n(-0.2)^{n+1} + (-0.2)^{n+1}}{(n+1)n} \leq \frac{1}{1000}$
$\dfrac{(-0.2)^{n+1}(n+1)}{(n+1)n} \leq \frac{1}{1000}$
$\dfrac{n}{(0.2)^{n+1}} \geq 1000$
$n \geq 1000 * (0.2)^{n+1}$
We graph or use numeric methods to see $n \geq 3$ (To solve for n we need to use the Lambert W function which is the inverse function of $f(W) = We^W$).

6. Max is 1, so n must be bigger than 5 since $\dfrac{1}{6!} = \dfrac{1}{720}$ but if $n = 6$, $\dfrac{1}{7!} = \dfrac{1}{5040}$

7.