Epidemic

An Introduction to Logistic Functions

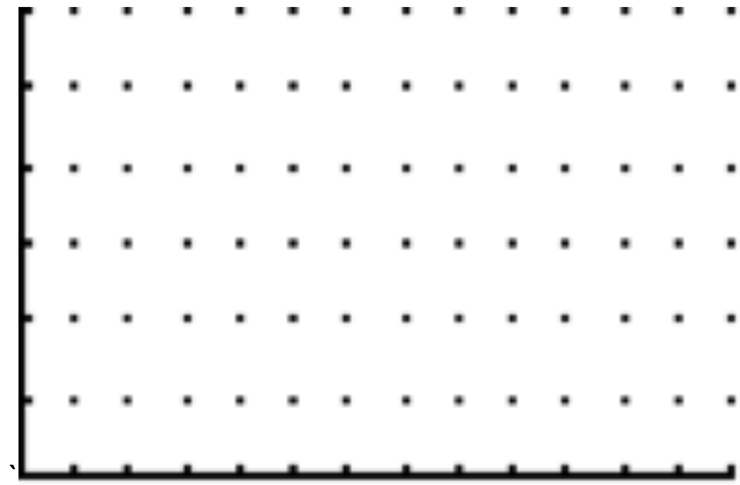
This exploration models the spread of a disease through a community with a finite population.

Introduction

Many times we attempt to model the behavior of an event through an artificial simulation. This exploration will mimic the spread of a disease such as a cold. The model assumes that a person with the disease will pass it on to two other people each day.

Procedure

1. Assign a number to each student in the class.
2. Using a TI-84 calculator, choose a student a random using the command **RandInt(1,N)** where N represents the number of students in the class.
3. The person selected is the index case. Have that person stand.
4. Keep track of the number of people standing after each iteration.
5. This demonstration assumes that each infected person can pass the disease to two other people each day. Press ENTER twice as many times as the number of people standing on the previous turn. Continue until all students are standing.
6. Create a scatter plot with the number of days required to infect all student in List1 and the total number of students infected through that day in List2. Manually set the window wide enough to see a few extra days on the right side.
7. Copy the results of your StatPlot onto the graph on the next page.



Part 1. Graphical Analysis

1. For which x-values does the graph seem to flatten out?

2. Circle the location on the plot where the graph seems to be growing fastest.

3. Using your calculator, find a good curve fit. Which model seems to fit best? Store your model in Y_1.

4. Graph your equation over the top of the data points. How well does the curve fit your data?

5. In Y_2 enter the command **Nderiv(Y_1,X,X)**. Graph Y_2 in the same window as the data plot and Y_1.

6. Use your calculator to find the coordinates of the relative maximum of the derivative. How does the x-coordinate compare to your results from question 2?

7. How does the y-coordinate of the model at the maximum of the derivative compare to the maximum value of the original data?

Part 2. Mathematical Analysis

As you saw in Part 1, the data starts out looking like an exponential function. If it were, it would follow the differential equation $\frac{dP}{dt} = kP.$ However, because there is a maximum population we introduce a limiting factor $(M - P),$ where M is the maximum carrying capacity of the local environment. This modifies the differential equation to $\frac{dP}{dt} = kP(M - P).$

Example 1.

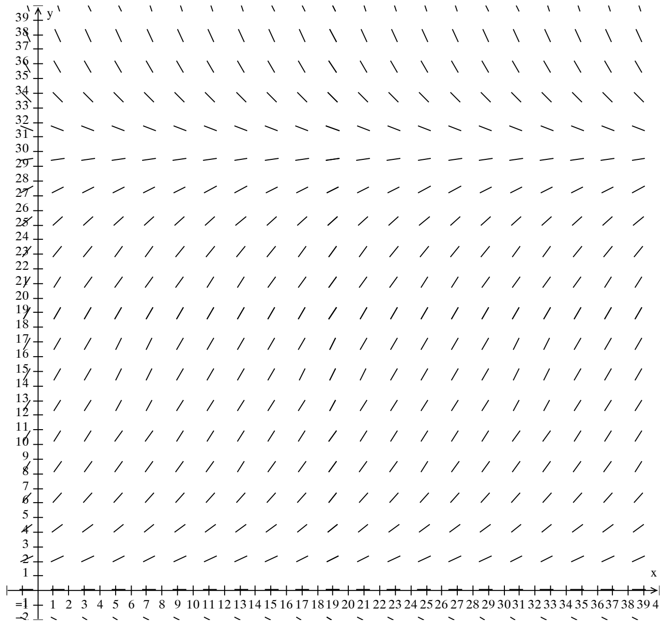
The growth rate of a population of mountain goats in a new game preserve is modeled by the differential equation $\frac{dP}{dt} = 0.008P(30 - P).$

a) What is the maximum carrying capacity for mountain goats in this game preserve?

b) What is the goat population when the population is growing fastest?

c) What is the rate of change of the goat population at this time?

d) The slope field below is the graph of the given differential equation. If 3 male and 3 female goats (6 total) were introduced when the game preserve started, sketch the graph of the number of goats over time.



e. Use separation of variables to solve the differential equation given that $P(0) = 6.$

Every logistic equation can be solved in a similar manner. It is easier to solve a general logistic differential equation with parameters for the constants to find a general formula.

The General Logistic Formula

The solution of the general logistic differential equation given by

Is

Where M is the carrying capacity and *k* is the growth constant (M and *k* are both positive). A is a constant determined by the initial conditions.

Example

$P(t) = \frac{\quad}{1+121.51e}$ where *t* is the number of months after some rabbits were released into the field.

a) Determine the maximum number of rabbits this field can maintain.

b) What is the value of *k*?

c) What is the population when $t = 0$?

d) What is the population when the rate of growth is greatest?

e) Use your calculator to determine when the population reaches the value found in part d).

Example 3. The rate at which the number of moose in a new section of a game preserve is changing is given by $\frac{dP}{dt} = 0.005P(40 - P)$ where *t* is given in years.

a) If 4 moose are introduced to this new habitat, what will be the maximum number of moose, the habitat will support?

b) When will the rate at which the moose population is growing be the greatest?

c) A well-meaning, but misinformed person, suggests that by introducing 50 moose initially, there will be more moose available to hunt. Explain why this is faulty reasoning.